JOURNAL OF APPROXIMATION THEORY **91**, 279–319 (1997)
ARTICLE NO. AT963061

On the Approximation Order of Principal Shift-Invariant Subspaces of $L_p(\mathbb{R}^d)$

Michael J. Johnson

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706, U.S.A.

Communicated by Rong-Qing Jia

Received February 15, 1995; accepted in revised form May 3, 1996

For $1 \leqslant p \leqslant \infty$, sufficient conditions on the generators $\{\phi_h\}_{h>0}$ are given which ensure that the h-dilates of the shift-invariant space generated by ϕ_h provide L_p-approximation of order $k>0$. Examples where ϕ_h is an exponential box spline or certain dilates of the Gaussian $e^{-|\cdot|^2}$ are considered; it is shown that our sufficient condition then provides an optimal lower bound on their approximation order.

1. INTRODUCTION

Let $d \in \mathbb{N} := \{1, 2, ...\}$, and let $C := (-1/2 \cdots 1/2)^d$ denote the open unit cube in $\mathbb{R}^d$. Following [19], define

$$\mathscr{L}_p := \left\{ f \in L_p(\mathbb{R}^d) \colon \|f\|_{\mathscr{L}_p} := \left\| \sum_{j \in \mathbb{Z}^d} |f(\cdot - j)| \right\|_{L_p(C)} < \infty \right\}, \qquad 1 \leqslant p \leqslant \infty.$$

Note that $\|f\|_{L_1} = \|f\|_{\mathscr{L}_1} \leqslant \|f\|_{\mathscr{L}_p} \leqslant \|f\|_{\mathscr{L}_{\bar{p}}}$ whenever $1 \leqslant p \leqslant \bar{p} \leqslant \infty$. It was shown in [19] that if $\phi \in \mathscr{L}_p$, then the semi-discrete convolution operator $\phi *'$ is a bounded operator from l_p into L_p, where $\phi *' c := \phi *'_1 c$ and

$$\phi *'_h c := \sum_{j \in \mathbb{Z}^d} c(hj)\, \phi(\cdot/h - j), \qquad h > 0.$$

We define $S_p(\phi)$ to be the image of $\phi *'$ on l_p:

$$S_p(\phi) := \left\{ \sum_{j \in \mathbf{Z}^d} c(j)\, \phi(\cdot - j) \colon c \in l_p \right\}.$$

$S_p(\phi)$ is said to be a *shift-invariant space* because $f(\cdot - j) \in S_p(\phi)$ whenever $f \in S_p(\phi)$ and $j \in \mathbb{Z}^d$. Since $S_p(\phi)$ is "generated" by the shifts (i.e., integer translates) of a single function, we call it a *principal* shift-invariant space.

0021-9045/97 $25.00

There are, in the literature, a number of ways of "generating" a shift-invariant space from a single function ϕ or a collection of functions Φ. When the details are unimportant we will simply write $S(\phi)$ or $S(\Phi)$ to denote this space. Shift-invariant (SI) spaces and principal shift-invariant (PSI) spaces are important in many areas of approximation theory, including the study of multivariate splines, radial basis function theory, sampling theory, wavelets, and subdivision schemes.

We can dilate any PSI space $S(\phi)$ by the parameter $h>0$ to obtain

$$S^h(\phi) := \{f(\cdot/h) : f \in S(\phi)\}.$$

The directed family $(S^h(\phi))_h$ is called a *ladder* of PSI spaces. When one has in hand a ladder of PSI spaces $(S^h(\phi))_h$, a standard problem, and one which has received considerable attention in the literature, is the determination of its L_p-approximation power, i.e., the determination of the rate of decay of $\operatorname{dist}(f, S^h(\phi); L_p)$ (as $h \to 0$) for sufficiently smooth $f \in L_p$. Here

$$\operatorname{dist}(f, A; X) := \inf_{x \in A} \|f - x\|_X.$$

In the literature, the statement, "$(S^h(\phi))_h$ provides L_p-approximation of order γ" has various definitions[1]; the essential ingredient is that

$$\operatorname{dist}(f, S^h(\phi); L_p) = O(h^\gamma) \qquad \text{for all sufficiently smooth} \quad f \in L_p. \tag{1.1}$$

Strang and Fix [34] have shown (see also [2, 12, 33]) that if ϕ is a compactly supported L_2 function, then the ladder $(S_2^h(\phi))_h$ provides "controlled" L_2-approximation of order $k \in \mathbb{N}$ if and only if $\hat{\phi}(0) \neq 0$ and one of the following two equivalent conditions holds:

$$\forall f \in \Pi_{k-1} \qquad \exists g \in \Pi_{k-1} \qquad \text{such that} \quad f = \phi *' g; \tag{1.2}$$

$$D^\alpha \hat{\phi}(j) = 0 \qquad \forall |\alpha| < k, \qquad j \in 2\pi \mathbb{Z}^d \backslash 0. \tag{1.3}$$

The qualifier "controlled," as used above, places a restriction on how the approximations to a smooth function can be drawn from $S_2^h(\phi)$ as $h \to 0$; hence, "controlled" approximation is stronger than unqualified approximation. Conditions (1.3) are now known as the Strang–Fix conditions of order k. These conditions had previously been considered for $d=1$ by Schoenberg [33]. Clarifications and extensions of [34] can be found in Dahmen and Micchelli [11] and Jia [17]. Finally, de Boor and Jia [7], using "local" rather than "controlled" approximation, extended the L_2 result of [34] to L_p for all $1 \leqslant p \leqslant \infty$. Later, interest in removing the compact support assumption on the generator ϕ developed and was

[1] Our definition will be stated in Section 2.

investigated by Jackson [16], Buhmann [10], Light and Cheney [25], Jia and Lei [18], and Halton and Light [15]. The following was proved in [18].

THEOREM 1.4. *Let $\phi \in L_\infty$ satisfy for some $\delta > 0$ and $k \in \mathbb{N}$*

$$\text{(i)} \quad |\phi(x)| = O(|x|^{-(d+k+\delta)}) \qquad \text{as} \quad |x| \to \infty;$$

$$\text{(ii)} \quad \phi(x) = \lim_{\varepsilon \to 0} \varepsilon^{-d} \int_{x+\varepsilon C} \phi \qquad \text{for all} \quad x \in \mathbb{R}^d.$$

Then, for $1 \leqslant p \leqslant \infty$, the ladder $(S_p^h(\phi))_h$ provides "controlled" L_p-approximation of order k if and only of $\hat{\phi}(0) \neq 0$ and the Strang–Fix conditions of order k are satisfied.

Here the "control" is a combination of that used in [34] and the "localness" used in [7]. Note that the compact support assumption of [7, 11, 34] has been replaced by the decay assumption (i) which, incidentally, becomes stronger as the approximation order k increases. All of the above-mentioned papers employ a technique known as quasi-interpolation/polynomial reproduction for their error analysis. Note that polynomial reproduction, as described in (1.2), requires that $\phi *' g$ be well defined for $g \in \Pi_{k-1}$, and hence the need for something like condition (i). In 1991, de Boor and Ron [8] were able to completely overcome condition (i) by performing their error analysis entirely in the Fourier-transformed domain. Moreover, their results applied to a more general situation which we now describe.

The ladder $(S^h(\phi))_h$ is known as a *stationary ladder* of PSI spaces because it is obtained by dilating the same PSI space $S(\phi)$. More generally we may use, as the h-entry of our ladder, the h-dilate of an h-dependent PSI space $S(\phi_h)$ to obtain a *non-stationary ladder* $(S^h(\phi_h))_h$. While in the stationary case properties of the ladder are hoped to be analyzed in terms of corresponding properties of the single generator ϕ, we need, in the non-stationary case, to inspect the entire family of generators $(\phi_h)_h$.

We can now state a sample from [8].

THEOREM 1.5. *Let $(\phi_h)_{h \in (0..h_0]}$ be a family of functions in $\mathscr{L}_\infty$ which satisfy $\hat{\phi} \neq 0$ on all of δC for some $\delta > 0$. If*

$$\sup_{h \in (0..h_0]} \sum_{j \in \mathbb{Z}^d \backslash 0} \left\| \frac{\hat{\phi}_h(\cdot + 2\pi j)}{(h^k + |\cdot|^k)\, \hat{\phi}_h} \right\|_{L_\infty(\delta C)} < \infty,$$

then $(S_\infty^h(\phi_h))_h$ provides L_∞-approximation of order k.

Proof. [8; Section 2.5].

Note that the only decay assumption imposed on ϕ_h is the mild assumption $\phi_h \in \mathscr{L}_\infty$. Following this result, de Boor *et al.* [4] considered the case $p=2$ where they were able to give a complete characterization of closed SI subspaces of L_2 which provide L_2-approximation of order $k>0$. Their results apply to non-stationary ladders of SI spaces and they make no decay assumptions on the generators. Kyriazis [21], in turn, considered stationary PSI spaces for the case $1<p<\infty$. Sufficient conditions on the generator $\phi \in L_p$ are given which, when satisfied, ensure that the stationary ladder $(S^h(\phi))_h$ provides L_p-approximation of order $k>0$. Again, no explicit decay assumptions are made on the generator ϕ.

In the present paper, we are concerned with providing lower bounds on the L_p-approximation order ($1 \leqslant p \leqslant \infty$) of non-stationary ladders of PSI spaces under the mild decay assumption that the generators belong to $\mathscr{L}_p$. An outline is as follows:

In Section 2, we define our notion of L_p-approximation order, and we state our main results. The proofs of these results comprise Section 5 and Section 6. These results are applied to non-stationary ladders of PSI spaces generated by exponential box splines and dilates of the Gaussian in Section 3 and Section 4, respectively. The particularly long proof of a Proposition in Section 3 is postponed until Section 8. In Section 7, side conditions are given under which the Strang–Fix conditions of order k are sufficient to ensure that the stationary ladder $(S_p^h(\phi))_h$ provides L_p-approximation of order k.

Throughout this paper, $|x| := |x|_2$ denotes the Euclidean norm of $x \in \mathbb{R}^d$ while for multi-indices $\alpha \in \{0, 1, 2, ...\}^d$, $|\alpha| := |\alpha|_1 := \sum_{i=1}^d |\alpha_i|$. For open $\Omega \subseteq \mathbb{R}^d$, $1 \leqslant p \leqslant \infty$, and $m \in \mathbb{Z}_+ := \{0, 1, 2, ...\}$ the Sobolev spaces $W_p^m(\Omega)$ are defined by

$$W_p^m(\Omega) := \left\{ f\colon \|f\|_{W_p^m(\Omega)} := \left(\sum_{|\alpha| \leqslant m} \|D^\alpha f\|_{L_p(\Omega)}^p \right)^{1/p} < \infty \right\},$$

with the usual modification when $p=\infty$. Corresponding to each $\alpha \in \mathbb{Z}_+^d$ is the power function $(\ )^\alpha\colon \mathbb{R}^d \to \mathbb{C}$ defined by

$$(\ )^\alpha\colon x \mapsto x^\alpha := \prod_{i=1}^d x(i)^{\alpha(i)}.$$

The space of polynomials of total degree at most k is denoted Π_k. The open unit ball in $\mathbb{R}^d$ is denoted by $B := \{x \in \mathbb{R}^d\colon |x| < 1\}$. For $f \in L_1 := L_1(\mathbb{R}^d)$, we denote its Fourier transform by

$$\hat{f}(x) := \int_{\mathbb{R}^d} e_{-x}(t)\, f(t)\, dt,$$

where e_x denotes the complex exponential given by

$$e_x(t) := e^{ix \cdot t}.$$

The Fourier transform extends by duality to the space of tempered distributions. The inverse Fourier transform of a tempered distribution f is denoted $f^\vee$. The collection of compactly supported $C^\infty(\mathbb{R}^d)$ functions is denoted by C_c^∞ and their Fourier transforms by $\widehat{C_c^\infty}$. All derivates of functions are to be understood as distributional. We use the symbol const to denote generic constants. It always denotes a real value in the interval $(0..\infty)$ and depends only on its arguments. Its value may change with each occurrence. When using the scaling parameter h as in $(S^h(\phi_h))_h$, it is assumed without further mention that $h \in (0..h_0]$ and $h_0 \in (0..1]$.

2. THE MAIN RESULTS

In order to make precise the notion "L_p-approximation of order γ," we need to specify which functions $f \in L_p$ are sufficiently smooth. This will be the Besov space $B_p^{\gamma,1}$ which we now define. Let $\eta \in \widehat{C_c^\infty}$ satisfy $\hat{\eta} = 1$ on a neighorhood of the origin, and for tempered distributions f, define

$$f_k := \begin{cases} (\hat{\eta}(2\cdot)\,\hat{f})^\vee, & \text{if} \quad k=0, \\ ((\hat{n}(2^{1-k}\cdot) - \hat{\eta}(2^{2-k}\cdot))\,\hat{f})^\vee, & \text{if} \quad k>0. \end{cases} \tag{2.1}$$

For $1 \leqslant p \leqslant \infty$, $\gamma \geqslant 0$, $1 \leqslant q \leqslant \infty$, the Besov space $B_p^{\gamma,q}$ (see [26]) can be defined as the collection of all tempered distributions f for which

$$\|f\|_{B_p^{\gamma,q}(\eta)} := \left(\sum_{k=0}^{\infty} 2^{\gamma k} \, \|f_k\|_{L_p}^q \right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$. It is known that $B_p^{\gamma,q}$ is a Banach space and, as such, is independent of the choice of η (i.e., different choices of η yield equivalent norms). We mention the following continuous imbeddings (cf. [26, p. 62]),

$$\begin{aligned} &B_p^{\gamma,q} \hookrightarrow B_p^{\gamma_1,q_1}, && \text{if} \quad \gamma_1 < \gamma \quad \text{or} \quad \gamma_1 = \gamma, \quad q_1 \geqslant q; \\ &B_p^{k,1} \hookrightarrow W_p^k(\mathbb{R}^d) \hookrightarrow B_p^{k,\infty}, && \text{if} \quad k \in \mathbb{Z}_+; \\ &B_p^{\gamma,1} \hookrightarrow \mathscr{H}_p^\gamma \hookrightarrow B_p^{\gamma,\infty}, && \text{if} \quad 1 < p < \infty, \end{aligned}$$

where $\mathscr{H}_p^\gamma$ is the potential space normed by

$$\|f\|_{\mathscr{H}_p^\gamma} := \|((1+|\cdot|^2)^{\gamma/2}\,\hat{f})^\vee\|_{L_p}, \qquad \gamma \geqslant 0, \qquad 1 < p < \infty.$$

DEFINITION 2.2. Let $1 \leqslant p \leqslant \infty$ and let $(\phi_h)_{h \in (0..h_0]}$ be a family in $\mathscr{L}_p$. We say that the ladder $(S_p^h(\phi_h))_h$ provides *L_p-approximation of order $\gamma > 0$* if there exists $c < \infty$ such that

$$\operatorname{dist}(f, S_p^h(\phi_h); L_p) \leqslant ch^\gamma \, \|f\|_{B_p^{\gamma,1}(\eta)}, \qquad \forall h \in (0..h_0], \qquad f \in B_p^{\gamma,1}.$$

We mention that it is a straightforward matter to show that if $(S_p^h(\phi_h))_h$ provides L_p-approximation of order γ, then

$$\operatorname{dist}(f, S_p^h(\phi_h); L_p) = O(h^\lambda) \qquad \text{as} \quad h \to 0 \qquad \forall f \in B_p^{\lambda,\infty}, \qquad 0 < \lambda < \gamma$$

$$\operatorname{dist}(f, S_p^h(\phi_h); L_p) = O(h^\gamma \, |\log h|) \qquad \text{as} \quad h \to 0 \qquad \forall f \in B_p^{\gamma,\infty}.$$

Throughout the remainder of this section, the exponent $\bar{p}$ will lie in the range $1 \leqslant \bar{p} \leqslant \infty$, the family of functions $(\phi_h)_{h \in (0..h_0]}$ will belong to $\mathscr{L}_{\bar{p}}$, δ will lie in $(0..2\pi)$, and η will be a function in $\widehat{C_c^\infty}$ which satifies

$$\operatorname{supp} \hat{\eta} \subset \delta C \qquad \text{and} \qquad \hat{\eta} = 1 \qquad \text{on} \quad \tfrac{1}{2}\delta C,$$

where $C = (-\frac{1}{2} .. \frac{1}{2})^d$.

The result which forms the foundation of the present paper is the following:

THEOREM 2.3. *If*

$$\sup_{0 < r \leqslant h} \operatorname{dist}(\eta, S_1^h(\phi_r); \mathscr{L}_{\bar{p}}) = O(h^\gamma) \qquad \textit{as} \quad h \to 0, \tag{2.4}$$

then $(S_p^h(\phi_h))_h$ provides L_p-approximation of order γ for all $1 \leqslant p \leqslant \bar{p}$.

Proof. cf. Section 6.

Note that in the stationary case, a necessary condition for $(S_p^h(\phi))_h$ to provide L_p-approximation of order γ is that

$$\operatorname{dist}(\eta, S_p^h(\phi); L_p) = O(h^\gamma) \qquad \text{as} \quad h \to 0. \tag{2.5}$$

Theorem 2.3 says that a condition slightly stronger than (2.5) (in fact, identical when $p = 1$) is actually sufficient:

$$\operatorname{dist}(\eta, S_1^h(\phi); \mathscr{L}_p) = O(h^\gamma) \qquad \text{as} \quad h \to 0. \tag{2.6}$$

In the non-stationary case, condition (2.6) suffices provided that it is equipped with a certain downward uniformity as described in (2.4). Once (2.4) has been established, the fact that we then obtain L_p-approximation orders for all $1 \leqslant p \leqslant \bar{p}$ is a simple consequence of the fact that $\|\cdot\|_{\mathscr{L}_p} \leqslant \|\cdot\|_{\mathscr{L}_{\bar{p}}}$.

With Theorem 2.3 in hand the job of establishing lower bounds on the L_p-approximation order of $(S_p^h(\phi_h))_h$, $1 \leqslant p \leqslant \bar{p}$, can be performed by estimating the ability of $S_1^h(\phi_r)$ to approximate η in $\mathscr{L}_{\bar{p}}$. One means for this is to choose $s \in S_1^h(\phi_r)$ so that $\hat{s} = \hat{\eta}$ on $2\pi C/h$, and then conclude that

$$\operatorname{dist}(\eta, S_1^h(\phi_r); \mathscr{L}_{\bar{p}}) \leqslant \|\eta - s\|_{\mathscr{L}_{\bar{p}}}. \tag{2.7}$$

This approach yields the following estimates:

PROPOSITION 2.8. *Assume that* $\hat{\phi}_h(h_0 \cdot) \neq 0$ *on all of* δC, $\forall 0 < h < h_0$. *Then for* $0 < r \leqslant k \leqslant h_0$,

$$(1) \quad \operatorname{dist}(\eta, S_1^h(\phi_r); \mathscr{L}_{\bar{p}}) \leqslant \left\| \left(\hat{\phi}_r(h \cdot) \sum_{j \in \mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\cdot + 2\pi j/h)}{\hat{\phi}_r(h \cdot + 2\pi j)} \right)^{\vee} \right\|_{\mathscr{L}_{\bar{p}}};$$

$$(2) \quad \operatorname{dist}(\eta, S_1^h(\phi_r); \mathscr{L}_{\bar{p}}) \leqslant \sum_{j \in \mathbb{Z}^d \setminus 0} \left\| \left(\frac{\hat{\eta} \hat{\phi}_r(h \cdot + 2\pi j)}{\hat{\phi}_r(h \cdot)} \right)^{\vee} \right\|_{\mathscr{L}_{\bar{p}}};$$

$$(3) \quad \operatorname{dist}(\eta, S_1^h(\phi_r); \mathscr{L}_{\bar{p}}) \leqslant \operatorname{const}(d, \eta, \bar{p}) \left(\sum_{j \in \mathbb{Z}^d \setminus 0} \left\| \frac{\hat{\phi}_r(h \cdot + 2\pi j)}{\hat{\phi}_r(h \cdot)} \right\|_{W_{\bar{q}}^m(\delta C)}^{\bar{q}} \right)^{1/\bar{q}},$$

where (3) *holds if* $2 \leqslant \bar{p} \leqslant \infty$ *in which case* $\bar{q}$ *is the exponent conjugate to* $\bar{p}$ (*i.e., satisfying* $1/\bar{p} + 1/\bar{q} = 1$) *and* m *is the least integer satisfying* $m > d/\bar{q}$.

Proof. cf. Section 6.

The proposition is intended to be used in conjunction with Theorem 2.3. The estimate (1) is actually a rewording of (2.7). The estimate (2) derives from (1) simply by pulling the summation outside of the norm. (3) derives from (1) using the crude estimate

$$\|g\|_{\mathscr{L}_{\bar{p}}} \leqslant \operatorname{const}(d, \eta, \bar{p}) \, \|\bar{g}\|_{W_{\bar{q}}^m(\mathbb{R}^d)}.$$

In Section 4 we will use Theorem 2.3 in conjunction with Proposition 2.8 (2) to invetigate the approximation order of non-stationary ladders generated by dilates of the Gaussian $e^{-|\cdot|^2}$. In Section 7, we apply Theorem 2.3 in conjunction with Proposition 2.8 (1) to show that in the stationary case, under certain side conditions, the Strang–Fix conditions of order k are sufficient to obtain approximation of order k. Here is a sample.

THEOREM 2.9. *Let* $\phi \in \mathscr{L}_\infty$ *satisfy* $\hat{\phi} \in C^{d+1}(\delta C)$ *and* $\hat{\phi} \in W_1^{d+k}(\delta C + 2\pi \mathbb{Z}^d \setminus 0)$. *If* $\hat{\phi}(0) \neq 0$ *and* ϕ *satisfies the Srang–Fix conditions of order* k (1.3), *then the stationary ladder* $(S_p^h(\phi))_h$ *provides* L_p*-approximation of order* k *for all* $1 \leqslant p \leqslant \infty$.

Proof. cf. Section 7.

An alternative means for estimating the $\mathcal{L}_{\bar{p}}$ distance between η and $S_1^h(\phi_r)$ is to take existing results for convergence in $L_{\bar{p}}$ and then show that when the approximand is η, the convergence is actually in $\mathcal{L}_{\bar{p}}$. Since $\eta \in \widehat{C_c^\infty}$ decays rapidly it is not surprising that the $L_{\bar{p}}$ convergence can be lifted to $\mathcal{L}_{\bar{p}}$ if the approximation scheme is sufficiently local (condition (i) below).

THEOREM 2.10. *Assume that there exists $\psi_h \in S_1(\phi_h)$, $h \in (0..h_0]$, such that for some $N \in \mathbb{N}$ and $c_1, c_2 < \infty$,*

(i) $|\psi_h(x)| \leqslant c_1(1+|x|)^{-(d+\gamma)}, \qquad \forall x \in \mathbb{R}^d, \qquad h \in (0..h_0];$

(ii) $\|f - \psi_r *_h' f\|_{L_{\bar{p}}} \leqslant c_2 \|f\|_N h^\gamma, \qquad \forall 0 < r \leqslant h \leqslant h_0, f \in C_c^\infty,$

where $\|f\|_N := \max_{|\alpha| \leqslant N} \max_{x \in \mathbb{R}^d} (1+|x|^2)^N |(D^\alpha f)(x)|.$

Then

$$\sup_{0<r\leqslant h} \operatorname{dist}(\eta, S_1^h(\phi_r); \mathcal{L}_{\bar{p}}) = O(h^\gamma) \qquad \text{as} \quad h \to 0.$$

Hence, by Theorem 2.3, $(S_p^h(\phi_h))_h$ provides L_p-approximation of order γ for all $1 \leqslant p \leqslant \bar{p}$.

Proof. cf. Section 6.

By employing an error analysis like that of [8] in order to verify condition (ii) of Theorem 2.10, we obtain the following result (compare with Theorem 1.4).

THEOREM 2.11. *Let $2 \leqslant \bar{p} \leqslant \infty$ and let $\bar{q}$ be the exponent conjugate to $\bar{p}$ (i.e., satisfying $1/\bar{p} + 1/\bar{q} = 1$). If there exist $c, \varepsilon \in (0..\infty)$ such that*

(i) $|\phi_h(x)| \leqslant c(1+|x|)^{-(d+\lceil\gamma\rceil+\varepsilon)}, \qquad \forall x \in \mathbb{R}^d, \qquad h \in (0..h_0];$

(ii) $\inf_{h \in (0..h_0]} |\hat{\phi}_h(0)| > 0;$

(iii) $A(\delta, \gamma, \bar{q}) := \sup_{h \in (0..h_0]} \left(\sum_{j \in \mathbb{Z}^d \setminus 0} \left\| \frac{\hat{\phi}_h(\cdot + 2\pi j)}{h^\gamma + |\cdot|^\gamma} \right\|_{L_\infty(\delta C)}^{\bar{q}} \right)^{1/\bar{q}} < \infty,$

then $(S_p^h(\phi_h))_h$ provides L_p-approximation of order γ for all $1 \leqslant p \leqslant \bar{p}$.

Here, $\lceil\gamma\rceil$ denotes the least integer greater or equal to γ. In Section 3, using Theorem 2.10 and Theorem 2.11 as well as results from [20, 28, 31], we will determine exactly the L_p-approximation order of exponential box splines for $1 \leqslant p \leqslant \infty$.

3. EXPONENTIAL BOX SPLINES

EXAMPLE 3.1. Let Ξ be a multiset of directions in $\mathbb{R}^d \backslash 0$ whose span covers all of $\mathbb{R}^d$, and let $\lambda \in \mathbb{C}^{\Xi}$. The family of exponential box splines ϕ_h, $h \geqslant 0$, is then defined by

$$\hat{\phi}_h := \prod_{\xi \in \Xi} \omega_\xi^h, \qquad \text{where} \quad \omega_\xi^h(x) := \int_0^1 e^{(h\lambda_\xi - i\xi \cdot x)\, t}\, dt. \tag{3.2}$$

We will show that for all $1 \leqslant p \leqslant \infty$, the L_p-approximation order of $(S_p^h(\phi_h))_h$ is exactly k' defined by

$$K_j := \{\xi \in \Xi : \xi \cdot j \in \mathbb{Z} \backslash 0\}, \qquad j \in \mathbb{Z}^d \backslash 0;$$
$$k' := \min\{\# K_j : j \in \mathbb{Z}^d \backslash 0\}.$$

For a general reference on box splines, the reader is referred to [6]. Actually, most of the claim in 3.1 is already known in its essence (i.e., in the sense of 1.1). The case when Ξ is confined to integral directions and $\lambda = 0$ has been settled in the work of [5]. The works of [13, 24, 27] treat the case of integral Ξ and general λ. For $p = \infty$, [28, 31] have settled the case of general Ξ and $\lambda = 0$. Reference [8], also working with $p = \infty$, established the upper bound on the approximation order for general Ξ and general λ. They provided the lower bound in case ϕ_0 was sufficiently smooth and the directions in Ξ were rational (while λ is still general). Reference [30] considers rational Ξ and general λ. For $p = 2$, both the lower bound and the upper bound are established. The lower bound on the approximation order is established for $2 < p \leqslant \infty$ excepting that in case $p = \infty$ it is required that $\hat{\phi}_0 \in L_1$. Reference [20] established the upper bound on the approximation order for general Ξ and general λ for $1 \leqslant p \leqslant \infty$. After completing the work on this example, I learned that Kyriazis [22] has extended the techniques of [21] to include some non-stationary ladders of PSI spaces. For $1 < p < \infty$, he establises the lower bound for rational Ξ and general λ under the assumption that $\hat{\phi}_0 \in L_1$.

The remainder of this section is devoted to proving the claim in 3.1. Since we are assuming that the directions in Ξ span $\mathbb{R}^d$ it follows that ϕ_h is a piecewise-exponential polynomial function supported in $\Xi[0..1]^m$, where $m := \# \Xi$ (cf. [27]). Aslo, as a distribution, ϕ_h has the following representation:

$$\int_{\mathbb{R}^d} \phi_h f\, dm = \int_{[0..1]^m} e^{h\lambda \cdot t} f(\Xi t)\, dt, \qquad f \in C_c^\infty.$$

It was shown in [20] that for all $f \in \widehat{C_c^\infty} \backslash 0$,

$$\operatorname{dist}(f, \mathscr{S}_p^h(\phi_h); L_p) \neq o(h^{k'}) \qquad \text{as} \quad h \to 0,$$

where $\mathscr{S}_p(\phi_h)$ is defined in [20]. For $1 \leqslant p < \infty$, $\mathscr{S}_p(\phi_h)$ is defined to be the closure in L_p of the finite linear span of the integer shifts of ϕ_h. Consequently, $S_p(\phi_h)$ is contained in $\mathscr{S}_p(\phi_h)$ for $1 \leqslant p < \infty$. For $p = \infty$ it was shown [20; Proposition 2.2] that $S_\infty(\phi) \subset \mathscr{S}_\infty(\phi)$ whenever ϕ satisfies

$$\sum_{j \in \mathbb{Z}^d} \|\phi\|_{L_\infty(j+C)} < \infty. \tag{3.3}$$

Since ϕ_h is bounded and compactly supported, (3.3) holds and we conclude that for all $f \in \widehat{C_c^\infty} \backslash 0$ and $1 \leqslant p \leqslant \infty$,

$$\operatorname{dist}(f, S_p^h(\phi_h); L_p) \neq o(h^{k'}) \qquad \text{as} \quad h \to 0.$$

Thus we need only concern ourselves with the task of showing that $(S_p^h(\phi_h))_h$ provides L_p-approximation of order k' for all $1 \leqslant p \leqslant \infty$. Since this task is vacuous when $k' = 0$, we may assume that $k' > 0$.

LEMMA 3.4.

$$\|\phi_h - \phi_0\|_{\mathscr{L}_\infty} \leqslant \operatorname{const}(d, \lambda, \Xi)\, h, \qquad \forall h \in [0..h_0].$$

Proof. Since $\operatorname{supp} \phi_h \subset \Xi[0..1]^m$ for all $h \geqslant 0$, it suffices to show that

$$\|\phi_h - \phi_0\|_{L_\infty} \leqslant \operatorname{const}(d, \lambda, \Xi)\, h, \qquad \forall h \in [0..h_0].$$

Recall that for any piecewise continuous function g,

$$\|g\|_{L_\infty} = \sup \left\{ \left| \int_{\mathbb{R}^d} g f \, dm \right| : f \in C_c^\infty, f \geqslant 0 \text{ and } \|f\|_{L_1} = 1 \right\}.$$

So let $f \in C_c^\infty$ be such that $f \geqslant 0$ and $\|f\|_{L_1} = 1$. Then

$$\begin{aligned}
\left| \int_{\mathbb{R}^d} (\phi_h - \phi_0) f \, dm \right| &= \left| \int_{[0..1]^m} (e^{h\lambda \cdot t} - 1) f(\Xi t) \, dt \right| \\
&\leqslant \operatorname{const}(d, \lambda)\, h \int_{[0..1]^m} f(\Xi t) \, dt \\
&\leqslant \operatorname{const}(d, \lambda)\, h \, \|\phi_0\|_{L_\infty} \\
&= \operatorname{const}(d, \lambda, \Xi)\, h. \quad \blacksquare
\end{aligned}$$

Note that as a consequence of the above lemma, we have that $\|\phi_h\|_{L_\infty}$ is bounded independently of $h \in [0..h_0]$.

In order to consider first an easier case, assume (for the time being) that $k'=1$. We will be applying Theorem 2.10 so let $\bar{p}:=\infty$ and $\gamma:=1$. It is known (cf. [28, Theorem 2.8; 31] that

$$\|f-\phi_0 *'_h f\|_{L_\infty} \leqslant \text{const}(d, \Xi)\, h\, \|f\|_{W^1_\infty}, \qquad \forall f \in W^1_\infty. \tag{3.5}$$

Let $\psi_h := \phi_h$, $h \in [0..h_0]$. That $\psi_h \in S_1(\phi_h)$ is of course trivial, and since the functions ϕ_h as well as their supports are bounded independently of $h \in [0..h_0]$, it follows that condition (i) of Theorem 2.10 is satisfied. In order to verify condition (ii), let $f \in C_c^\infty$. Then, for $0 < r \leqslant h \leqslant h_0$,

$$\begin{aligned}
\|f-\psi_r *'_h f\|_{L_\infty} &\leqslant \|f-\phi_0 *'_h f\|_{L_\infty} + \|(\phi_0-\phi_r) *'_h f\|_{L_\infty}\\
&\leqslant \text{const}(d, \Xi)\, h\, \|f\|_{W^1_\infty} + \|\phi_0-\phi_r\|_{\mathscr{L}_\infty} \|f\|_{L_\infty},\\
&\qquad \text{by (3.5) and Lemma 5.1},\\
&\leqslant \text{const}(d, \lambda, \Xi)\, h\, \|f\|_1 \qquad \text{by Lemma 3.4 as} \quad r \leqslant h.
\end{aligned}$$

Thus condition (ii) of Theorem 2.10 is satsified with $N:=1$ and we conclude therefore that $(S^h_p(\phi_h))_h$ provides L_p-approximation of order $k'=1$ for all $1 \leqslant p \leqslant \infty$.

We turn now to the more difficult case $k'>1$ where we will apply Theorem 2.11 with $\gamma := k'$ and $\bar{p} := \infty$. It follows from (3.2) that there exists $\delta \in (0..\pi)$ and $h_0 \in (0..1]$, depending only on (d, λ, Ξ), such that

$$\begin{aligned}
|\hat{\phi}_h(x)| &> \delta, && \forall x \in \delta C, && h \in [0..h_0],\\
|h\lambda_\xi - i\xi \cdot x| &< 1 && \forall x \in \delta C, && h \in [0..h_0], \qquad \xi \in \Xi.
\end{aligned} \tag{3.6}$$

In particular, condition (ii) of Theorem 2.11 is satisfied. Since the functions ϕ_h as well as their supports are bounded independently of $h \in [0..h_0]$, condition (i) of Theorem 2.11 is satisfied.

In showing that condition (iii) of Theorem 2.11 is satisfied, we will be following the approach taken in the Box Spline section of [8]. There, $A(\delta, k', 1) < \infty$ was established only when ϕ_0 was sufficiently smooth and Ξ was rational. Later, the sufficiently smooth aspect was identified [30] as being when $\hat{\phi}_0 \in L_1$. The following proposition can be used to show that $\hat{\phi}_0 \in L_1$ whenever $k'>1$.

PROPOSITION 3.7. *If $k'>1$, then*

$$\sum_{j \in \mathbb{Z}^d \backslash 0} \prod_{\xi \in \Xi} \frac{1}{1+|\xi \cdot j|} < \infty.$$

Proof. cf. Section 8.

The following lemma and its proof are taken almost directly from [8]. By placing $1+|\xi\cdot j|$ in the denominator of our estimate (instead of $|\xi\cdot j|$ as in [8, 30]) we get by without assuming Ξ to be rational.

LEMMA 3.8. *For all* $j\in\mathbb{Z}^d\backslash 0$, $h\in[0..h_0]$, *and* $x\in\delta C$,

$$|\omega_\xi^h(x+2\pi j)|\leqslant\frac{\mathrm{const}(d,\lambda,\Xi)}{1+|\xi\cdot j|}\begin{cases}1, & \xi\in\Xi\backslash K_j\\ h+|x|, & \xi\in K_j.\end{cases}$$

Proof. Fix $j\in\mathbb{Z}^d\backslash 0$, $\xi\in\Xi$, $h\in[0..h_0]$, and $x\in\delta C$. Note that

$$\omega_\xi^h(x+2\pi j)=\begin{cases}1, & \text{if}\quad h\lambda_\xi-i\xi\cdot x=2\pi i\xi\cdot j\\ \dfrac{e^{h\lambda_\xi-i\xi\cdot(x+2\pi j)}-1}{h\lambda_\xi-i\xi\cdot x-2\pi i\xi\cdot j}, & \text{otherwise.}\end{cases}$$

Also, by (3.2) and (3.6), $|\omega_\xi^h(x+2\pi j)|\leqslant e$.

Case 1. $\xi\in\Xi\backslash K_j$.

If $|\xi\cdot j|\leqslant 1$, then the lemma holds with $\mathrm{const}(d,\lambda,\Xi)\geqslant 2e$. If, on the other hand, $|\xi\cdot j|>1$, then $h\lambda_\xi-i\xi\cdot x\neq 2\pi i\xi\cdot j$ and hence

$$|\omega_\xi^h(x+2\pi j)|\leqslant\frac{e^{|h\lambda_\xi|}+1}{|2\pi\xi\cdot j|-1}\leqslant\frac{e+1}{(2\pi-2)\,|\xi\cdot j|+2\,|\xi\cdot j|-1}$$

$$\leqslant\frac{e+1}{(2\pi-2)\,|\xi\cdot j|+1}\leqslant\frac{\mathrm{const}(d,\lambda,\Xi)}{1+|\xi\cdot j|}.$$

Case 2. $\xi\in K_j$.

By (3.6), $h\lambda_\xi-i\xi\cdot x\neq 2\pi i\xi\cdot j$; hence,

$$|\omega_\xi^h(x+2\pi j)|\leqslant\frac{|e^{h\lambda_\xi-i\xi\cdot x}-1|}{2\pi\,|\xi\cdot j|-1}\leqslant\frac{\mathrm{const}(d,\lambda,\Xi)(h+|x|)}{(2\pi-2)\,|\xi\cdot j|+1}$$

$$\leqslant\frac{\mathrm{const}(d,\lambda,\Xi)(h+|x|)}{1+|\xi\cdot j|},$$

thus proving the lemma. ∎

Therefore, by (3.2) and Lemma 3.8,

$$|\hat{\phi}_h(x+2\pi j)| \leqslant \text{const}(d, \lambda, \Xi)(h+|x|)^{\# K_j} \prod_{\xi \in \Xi} \frac{1}{1+|\xi \cdot j|}$$

$$\leqslant \text{const}(d, \lambda, \Xi)(h^{k'}+|x|^{k'}) \prod_{\xi \in \Xi} \frac{1}{1+|\xi \cdot j|},$$

$$\forall x \in \delta C, \qquad j \in \mathbb{Z}^d \backslash 0.$$

Hence, with Proposition 3.7 in view,

$$A(\delta, k', 1) \leqslant \text{const}(d, \lambda, \Xi) \sum_{j \in \mathbb{Z}^d \backslash 0} \prod_{\xi \in \Xi} \frac{1}{1+|\xi \cdot j|} < \infty,$$

thus estabishing condition (iii) of Theorem 2.11. Therefore, by Theorem 2.11, $(S_p^h(\phi_h))_h$ provides L_p-approximation of order k' for all $1 \leqslant p \leqslant \infty$.

4. THE GAUSS KERNEL

EXAMPLE 4.1. For $h \in (0..1]$, define ϕ_h by

$$\hat{\phi}_h(x) := e^{-\mu(h)\,|x|^2/4\pi^2}, \qquad \text{where} \quad \mu(h) := \gamma \log(e/h)$$

for some $\gamma > 0$. We will show that the L_p-approximation order of $(S_p^h(\phi_h))_h$ is exactly γ for all $1 \leqslant p \leqslant \infty$.

That $(S_p^h(\phi_h))_h$ provides L_p-approximation of order γ (in, say, the sense of (1.1)) is known for $p = \infty$ and $p = 2$. For the precise details see [3; 8; Theorem 3.8] ($p = \infty$) and [29, Corollary 2.35] ($p = 2$). As for the upper bound on the approximation order, it was shown in [20] that there exists $f \in \widehat{C_c^\infty} \backslash 0$ such that

$$\text{dist}(f, \mathscr{S}_p^h(\phi_h); L_p) \neq o(h^\gamma) \qquad \text{as} \quad h \to 0,$$

where $\mathscr{S}_p(\phi_h)$ is defined in [20]. As mentioned in the discussion prior to (3.3), $S_p(\phi_h) \subset \mathscr{S}_p(\phi_h)$ for all $1 \leqslant p \leqslant \infty$ (as (3.3) holds in case $p = \infty$). Hence the L_p-approximation order of $(S_p^h(\phi_h))_h$ cannot exceed γ.

The task of showing that $(S_p^h(\phi_h))_h$ provides L_p-approximation of order γ is simplified by making use of the tensor product nature of ϕ_h and by employing the following:

LEMMA 4.2. *Let* $f_i \in \mathscr{L}_\infty(\mathbb{R})$, $i = 1, 2, ..., d$, *be continuous and define* $f(x) := f_1(x_1)\, f_2(x_2) \cdots f_d(x_d)$, $x \in \mathbb{R}^d$. *Then*

$$\|f\|_{\mathscr{L}_\infty(\mathbb{R}^d)} \leqslant \prod_{i=1}^{d} \|f_i\|_{\mathscr{L}_\infty(\mathbb{R})}.$$

Proof. The lemma is clear when $d = 1$. Proceeding by induction, assume the lemma to be true for $d' = d - 1$ and consider d. Let $x \in \mathbb{R}^d$. Then

$$\begin{aligned}
\sum_{j \in \mathbb{Z}^d} |f(x+j)| &= \sum_{k \in \mathbb{Z}^{d-1}} |f_1(x_1 + k_1)\, f_2(x_2 + k_2) \cdots f_{d-1}(x_{d-1} + k_{d-1})| \\
&\quad \times \sum_{n \in \mathbb{Z}} |f_d(x_d + n)| \\
&\leqslant \left(\prod_{i=1}^{d-1} \|f_i\|_{\mathscr{L}_\infty(\mathbb{R})} \right) \sum_{n \in \mathbb{Z}} |f_d(x_d + n)|, \\
&\qquad \text{by induction hypothesis,} \\
&\leqslant \left(\prod_{i=1}^{d} \|f_i\|_{\mathscr{L}_\infty(\mathbb{R})} \right)
\end{aligned}$$

which proves the lemma. ∎

Let $\tau \in C^\infty(\mathbb{R})$ be supported in $(-1..1)$ and be such that $\tau = 1$ on $[-1/2..1/2]$. Define $\hat{\eta}(x) := \tau(x_1)\, \tau(x_2) \cdots \tau(x_d)$, $x \in \mathbb{R}^d$. Note that $\eta \in \widehat{C_c^\infty}$, $\operatorname{supp} \hat{\eta} \subset 2C$, and $\hat{\eta} = 1$ on C. Now for $j \in \mathbb{Z}^d$,

$$\frac{\hat{\eta}(x)\, \hat{\phi}_r(hx + 2\pi j)}{\hat{\phi}_r(hx)} = \prod_{i=1}^{d} \tau(x_i)\, e^{-\mu(r)(hx_i j_i/\pi + j_i^2)}. \tag{4.3}$$

Define

$$a(k) := \|(\tau e^{-\mu(r)(k^2 + hk\cdot/\pi)})^\vee\|_{\mathscr{L}_\infty(\mathbb{R})}, \qquad k \in \mathbb{Z}.$$

Then

$$\begin{aligned}
\Gamma(r, h) &:= \sum_{j \in \mathbb{Z}^d \backslash 0} \left\| \left(\frac{\hat{\eta}\hat{\phi}_r(h\cdot + 2\pi j)}{\hat{\phi}_r(h\cdot)} \right)^\vee \right\|_{\mathscr{L}_\infty} \\
&\leqslant \sum_{j \in \mathbb{Z}^d \backslash 0} \prod_{i=1}^{d} a(j_i), \qquad \text{by Lemma 4.2 and (4.3),} \\
&\leqslant d \sum_{k \in \mathbb{Z} \backslash 0} a(k) \sum_{j \in \mathbb{Z}^{d-1}} \prod_{i=1}^{d-1} a(j_i) \\
&= d \left(\sum_{k \in \mathbb{Z}} a(k) \right)^{d-1} \sum_{k \in \mathbb{Z} \backslash 0} a(k).
\end{aligned} \tag{4.4}$$

By Lemma 5.2,

$$\|(\tau g)^{\vee}\|_{\mathscr{L}_\infty(\mathbb{R})} \leqslant \text{const}\ \|\tau g\|_{W_1^2(\mathbb{R})} \leqslant \text{const}(\tau)\ \|g\|_{W_1^2([-1..1])},$$
$$\forall g \in W_1^2([-1..1]).$$

In particular, for $k \in \mathbb{Z}\backslash 0$,

$$\frac{a(k)}{\text{const}(\tau)} \leqslant \|e^{-\mu(r)(k^2 + hk\cdot/\pi)}\|_{W_1^2([-1..1])}$$
$$= \|(1 + \mu(r)\, h\, |k|/\pi + (\mu(r)\, hk/\pi)^2)\, e^{-\mu(r)(k^2 + hk\cdot/\pi)}\|_{L_1([-1..1])}$$
$$\leqslant (2 + 1 + \mu(r)\, h\, |k|/\pi)\, e^{-\mu(r)(k^2 - h\,|k|/\pi)}.$$

Hence,

$$\sum_{k \in \mathbb{Z}\backslash 0} a(k) \leqslant \text{const}(\tau)(1 + \mu(r)\, h) \sum_{k=1}^{\infty} ke^{-\mu(r)(k^2 - hk/\pi)}$$
$$= \text{const}(\tau)(1 + \mu(r)\, h)\, e^{-\mu(r)(1 - h/\pi)} \sum_{k=1}^{\infty} ke^{-\mu(r)(k^2 - 1 - h(k-1)/\pi)}$$
$$\leqslant \text{const}(\tau)(1 + \mu(r)\, h)\, e^{-\mu(r)(1 - h/\pi)} \left(1 + \sum_{k=2}^{\infty} ke^{-\mu(r)\, k}\right)$$
$$\leqslant \text{const}(\tau)(1 + \mu(r)\, h)\, e^{-\mu(r)(1 - h/\pi)}, \qquad \text{since} \quad \mu(r) \geqslant 1.$$

Combining this with (4.4) yields

$$\Gamma(r, h) \leqslant d(\|\tau^{\vee}\|_{\mathscr{L}_\infty} + \text{const}(\tau)(1 + \mu(r)\, h)\, e^{-\mu(r)(1 - h/\pi)})^{d-1}$$
$$\times \text{const}(\tau)(1 + \mu(r)\, h)\, e^{-\mu(r)(1 - h/\pi)}.$$

Noting that $\mu(r)\, e^{-\mu(r)(1 - 1/\pi)}$ is bounded independently of r, we conclude that

$$\Gamma(r, h) \leqslant \text{const}(d, \tau)(1 + \mu(r)\, h)\, e^{-\mu(r)(1 - h/\pi)}. \tag{4.5}$$

Applying elementary differential calculus to (4.5), it can be shown that

$$\sup_{0 < r \leqslant h} \Gamma(r, h) \leqslant \text{const}(d, \tau, \gamma)\, h^{\gamma}, \qquad \forall 0 < h \leqslant 1.$$

Therefore, by Theorem 2.3 in conjugation with Proposition 2.8 (2), $(S_p^h(\phi_h))_h$ provides L_p-approximation or order γ for all $1 \leqslant p \leqslant \infty$.

5. SOME USEFUL LEMMATA

In this section we march out a few results which will be useful in proving our main results. At the outset of the Introduction, we mention a result of [19]. It can be stated in slightly more generality as follows:

LEMMA 5.1. *Let* $1 \leqslant p \leqslant \infty$ *and let* $\phi \in \mathscr{L}_p$. *Then*

$$\|\phi *'_h f\|_{L_p} \leqslant \|\phi\|_{\mathscr{L}_p} h^{d/p} \|f\|_{l_p(h\mathbb{Z}^d)}, \qquad \forall f \in l_p(h\mathbb{Z}^d).$$

Proof. See [19, Theorem 2.1] for the case $h=1$. The general case $h>0$ can now be derived from the fact that $\|g(\cdot/h)\|_{L_p} = h^{d/p} \|g\|_{L_p}$. ∎

The following lemma gives an estimate of the $\mathscr{L}_p$ norm of a function g in terms of $\hat{g}$ for $2 \leqslant p \leqslant \infty$.

LEMMA 5.2. *Let* $2 \leqslant p \leqslant \infty$ *and let* q *be the exponent conjugate to* p (*i.e., satisfying* $1/p + 1/q = 1$). *Let* m *be the least integer satisfying* $m > d/q$. *Then*

$$\|g\|_{\mathscr{L}_p} \leqslant \mathrm{const}(d,p) \|\hat{g}\|_{W_q^\infty(\mathbb{R}^d)}, \qquad \forall \hat{g} \in W_q^m(\mathbb{R}^d).$$

Proof. Let $\hat{g} \in W_q^m(\mathbb{R}^d)$. Then

$$\begin{aligned}
\|g\|_{\mathscr{L}_p} &\leqslant \sum_{j \in \mathbb{Z}^d} \|g\|_{L_p(j+C)} = \sum_{j \in \mathbb{Z}^d} (1+|j|)^{-m} (1+|j|)^m \|g\|_{L_p(j+C)} \\
&\leqslant \left(\sum_{j \in \mathbb{Z}^d} (1+|j|)^{-mq} \right)^{1/q} \|((1+|j|)^m \|g\|_{L_p(j+C)})_j\|_{l_p(\mathbb{Z}^d)}, \\
&\qquad \text{by Hölders inequality,} \\
&\leqslant \mathrm{const}(d,p) \|(\|(1+|\cdot|)^m g\|_{L_p(h+C)})_j\|_{l_p(\mathbb{Z}^d)} \\
&= \mathrm{const}(d,p) \|(1+|\cdot|)^m g\|_{L_p}.
\end{aligned} \tag{5.3}$$

By the Hausdorff–Young theorem (cf. [23, p. 142]),

$$\|f\|_{L_p} \leqslant \mathrm{const}(d) \|\hat{f}\|_{L_q}, \qquad \forall \hat{f} \in L_q. \tag{5.4}$$

Since $(-ix)^\alpha f(x) = \left(D^\alpha \hat{f}\right)^\vee (x)$ it is easy to extend (5.4) to obtain

$$\|(1+|\cdot|)^m f\|_{L_p} \leqslant \mathrm{const}(d,p) \|\hat{f}\|_{W_q^m(\mathbb{R}^d)}, \qquad \forall \hat{f} \in W_q^m(\mathbb{R}^d). \tag{5.5}$$

The lemma now follows by (5.3) and (5.5). ∎

The following lemma shows that the l_p norm of band-limited functions is dominated by their L_p norm. Actually, they are equivalent, but we need only this direction here.

LEMMA 5.6. *For all* $h > 0$ *and* $1 \leqslant p \leqslant \infty$,

$$h^{d/p} \|f\|_{l_p(h\mathbb{Z}^d)} \leqslant \mathrm{const}(d) \|f\|_{L_p(\mathbb{R}^d)},$$

whenever $f \in L_p$ *and* $\operatorname{supp} \hat{f} \subset h^{-1} 2\pi C$.

Here, we employ the slightly abusive notation

$$\|f\|_{l_p(h\mathbb{Z}^d)} := \|f_{|h\mathbb{Z}^d}\|_{l_p(h\mathbb{Z}^d)}.$$

Proof. It suffices to prove the lemma for the special case $h = 1$ since the general case $h > 0$ can then be obtained by scaling. For a proof when $h = 1$, see [14, Lemma 1]. ∎

The following lemmata show how the semi-discrete convolution acts in the Fourier-transformed domain.

LEMMA 5.7. *Let* $\phi \in \widehat{C_c^\infty}$, *and let* f *be a tempered distribution such that* $\operatorname{supp} \hat{f}$ *is compact. Then for all* $h > 0$,

$$(\phi *'_h f)^\wedge = \hat{\phi}(h\cdot) \sum_{j \in \mathbb{Z}^d} \hat{f}(\cdot - 2\pi j/h).$$

Proof. It suffices to prove the lemma for the case $h = 1$ since the general case $h > 0$ can then be obtained by scaling. We prove the lemma first for the special case $f \in \widehat{C_c^\infty}$. So assume temporarily that $f \in \widehat{C_c^\infty}$. We then have by Poisson's summation formula (cf. [35, Chap. 7])

$$\begin{aligned}\sum_{j \in \mathbb{Z}^d} \hat{f}(x - 2\pi j) &= \sum_{j \in \mathbb{Z}^d} (e_{-x} f)^\wedge (2\pi j) \\ &= \sum_{j \in \mathbb{Z}^d} e_{-x}(j)\, f(j) = \sum_{j \in \mathbb{Z}^d} f(j)\, e_{-j}(x), \qquad \forall x \in \mathbb{R}^d. \quad (5.8)\end{aligned}$$

Since $\phi \in \widehat{C_c^\infty} \subset L_1$ and $\sum_{j \in \mathbb{Z}^d} |f(j)| < \infty$,

$$\begin{aligned}(\phi *' f)^\wedge (x) &= \sum_{j \in \mathbb{Z}^d} f(j)(\phi(\cdot - j))^\wedge (x) = \sum_{j \in \mathbb{Z}^d} f(j)\, \hat{\phi}(x)\, e_{-j}(x) \\ &= \hat{\phi}(x) \sum_{j \in \mathbb{Z}^d} \hat{f}(x - 2\pi j), \qquad \text{by (5.8)},\end{aligned}$$

and thus proving the lemma for the special case $f \in \widehat{C_c^\infty}$. For the general case, let σ_n be a delta sequence in C_c^∞ (e.g., $\sigma_n := n^d \sigma(n\cdot)$ with $\sigma \in C_c^\infty$, $\sigma \geqslant 0$, and $\int \sigma = 1$). Put $f_n := \hat{\sigma}_n f$, $n \in \mathbb{N}$. Then since $f_n \in \widehat{C_c^\infty}$, we have that

$$(\phi *' f_n)^\wedge = \hat{\phi} \sum_{j \in \mathbb{Z}^d} \hat{f}_n(\cdot - 2\pi j), \qquad n \in \mathbb{N}.$$

Since supp $\hat{f}$ is compact, it follows (cf. [32, Theorem 6.32]) that $\hat{f}_n \to \hat{f}$ in the space of tempered distributions. Therefore,

$$\hat{\phi} \sum_{j \in \mathbb{Z}^d} \hat{f}_n(\cdot - 2\pi j) \to \hat{\phi} \sum_{j \in \mathbb{Z}^d} \hat{f}(\cdot - 2\pi j)$$

in the space of tempered distributions (as $\hat{\phi} \in C_c^\infty$ implies that sums can be taken over some finite subset of $\mathbb{Z}^d$). Thus, the lemma will be proved as soon as we show that

$$(\phi *' f_n)^\wedge \to (\phi *' f)^\wedge \tag{5.9}$$

in the space of tempered distributions. For that, note that since supp $\hat{f}$ is compact, there exists $N \in Z_+$ such that $|f(x)| = O(|x|^N)$ as $|x| \to \infty$ (cf. [32, Theorems 6.8, 7.23]). Hence,

$$\sup_{j \in \mathbb{Z}^d} |f(j) - f_n(j)| \, (1 + |j|)^{-(N+1)} \to 0 \qquad \text{as} \quad n \to \infty.$$

It now follows from the rapid decay of $\phi \in \widehat{C_c^\infty}$ that $\phi *' f_n \to \phi *' f$ in the space of tempered distributions. Therefore, (5.9) holds (cf. [32, Theorem 7.15]). ∎

The assumption that $\phi \in \widehat{C_c^\infty}$ above is too strong for most purposes. It can be relaxed provided that we further restrict f.

LEMMA 5.10. *Let $\phi \in L_1$. If $f \in L_1$ and* supp $\hat{f}$ *is compact, then for all* $h > 0$,

$$\|f\|_{l_1(h\mathbb{Z}^d)} < \infty;$$

$$(\phi *'_h f)^\wedge = \hat{\phi}(h\cdot) \sum_{j \in \mathbb{Z}^d} \hat{f}(\cdot - 2\pi j/h).$$

Proof. It suffices to prove the lemma for the case $h = 1$ since the general case $h > 0$ can then be obtained by scaling. Let $f \in L_1$ be such that $\hat{f}$ is of compact support. There exists a sufficiently large $n \in \mathbb{N}$ such that supp $\hat{f} \subset n\pi 2C$. Hence, by Lemma 5.6,

$$n^{-d} \|f\|_{l_1(n^{-1}\mathbb{Z}^d)} \leqslant M_0 \|f\|_{L_1(\mathbb{R}^d)}.$$

Since $\mathbb{Z}^d \subset n^{-1}\mathbb{Z}^d$, it follows that $\|f\|_{l_1(\mathbb{Z}^d)} < \infty$. Hence $\phi *' f \in L_1$ and

$$(\phi *' f)^\wedge = \hat{\phi} \sum_{j \in \mathbb{Z}^d} f(j)\, e_{-j} = \hat{\phi} \sum_{j \in \mathbb{Z}^d} f(-j)\, e_j.$$

It is now a straightforward matter to complete the proof by verifying that $f(-j)$ is in fact the jth Fourier coefficient of the $2\pi\mathbb{Z}^d$-periodic function $\sum_{j\in\mathbb{Z}^d}\hat{f}(\cdot-2\pi j)$. ∎

LEMMA 5.11 (Wiener's Lemma). *Let $f, g\in L_1$ be such that* supp $\hat{f}$ *is compact and $\hat{g}(x)\neq 0$ for all $x\in$* supp $\hat{f}$. *Then*

$$\left(\frac{\hat{f}}{\hat{g}}\right)^{\vee}\in L_1.$$

Proof. cf. [32, Theorem 11.6].

In the following lemma, a description is given for a multi-level approximation scheme employing the dilated shifts of a function η. In subsequent theorems, this approximation scheme will be used except that η will be replaced by a suitable approximation of η drawn from dilates of $S_1(\phi_h)$.

LEMMA 5.12. *Let $1\leqslant p\leqslant\infty$, and let $\eta\in\widehat{C_c^\infty}$ and $\delta\in(0\,..\,2\pi)$ be such that* supp $\hat{\eta}\subset\delta C$ *and $\hat{\eta}=1$ on $\frac{1}{2}\delta C$. For $h\in(0\,..\,1]$, let $n:=n(h)$ be the largest integer for which $2^n h\leqslant 1$. Let $\gamma>0$. For $f\in B_p^{\gamma,1}$, let $\{f_k\}_{k\in\mathbb{Z}_+}$ be as in* (2.1). *Then for all $h\in(0\,..\,1]$*

(1) $f_k=\eta *'_{h2^{n-k}} f_k,\qquad \forall k\in\mathbb{Z}_+;$

(2) $(h2^{n-k})^{d/p}\,\|f_k\|_{l_p(h2^{n-k}\mathbb{Z}^d)}\leqslant\text{const}(d)\,\|f_k\|_{L_p},\qquad\forall k\in\mathbb{Z}_+;$

(3) $\left\|f-\sum_{k=0}^{n} f_k\right\|_{L_p}\leqslant h^\gamma\,\|f\|_{B_p^{\gamma,1}(\eta)}.$

Proof. Note that supp $\hat{f}_k$ is compact. Hence, by Lemma 5.7,

$$(\eta *'_{h2^{n-k}} f_k)^\wedge=\hat{\eta}(h2^{n-k}\cdot)\sum_{j\in\mathbb{Z}^d}\hat{f}_k(\cdot-2\pi j/(h2^{n-k})).$$

by (2.1), supp $\hat{f}_k\subseteq$ supp $\hat{\eta}(2^{1-k}\cdot)\subseteq 2^{k-1}\varepsilon C$, $\forall k\in\mathbb{Z}_+$. It is now a straightforward matter to verify that $\hat{\eta}(h2^{n-k}\cdot)$ and $\hat{f}_k(\cdot-2\pi j/(h2^{n-k}))$ have disjoint supports whenever $j\in\mathbb{Z}^d\backslash 0$ and that $\hat{\eta}(h2^{n-k}\cdot)=1$ on the support of $\hat{f}_k$. Therefore, $(n *'_{h2^{n-k}} f_k)^\wedge=\hat{f}_k$ which proves (1). Now,

$$\text{supp}(f_k(h2^{n-k}\cdot))^\wedge\subseteq h2^{n-k}2^{k-1}\varepsilon C\subset 2\pi C.$$

Hence, by Lemma 5.6,

$$\begin{aligned}\|f_k\|_{l_p(h2^{n-k}\mathbb{Z}^d)}&=\|f_k(h2^{n-k}\cdot)\|_{l_p(\mathbb{Z}^d)}\leqslant\text{const}(d)\,\|f_k(h2^{n-k}\cdot)\|_{L_p}\\&=\text{const}(d)(h2^{n-k})^{-d/p}\,\|f_k\|_{L_p}.\end{aligned}$$

Hence (2). In order to verify (3), note that

$$\hat{f} - \sum_{k=0}^{n} \hat{f}_k = \left(\hat{\eta}(2\cdot) + \sum_{k=1}^{\infty} (\hat{\eta}(2^{1-k}\cdot) - \hat{\eta}(2^{2-k}\cdot)) \right) \hat{f}$$

$$- \left(\hat{\eta}(2\cdot) + \sum_{k=1}^{n} (\hat{\eta}(2^{1-k}\cdot) - \hat{\eta}(2^{2-k}\cdot)) \right) \hat{f}$$

$$= \hat{f} \sum_{k=n+1}^{\infty} (\hat{\eta}(2^{1-k}\cdot) - \hat{\eta}(2^{2-k}\cdot)).$$

Therefore,

$$\left\| f - \sum_{k=0}^{n} f_k \right\|_{L_p} \leqslant \sum_{k=n+1}^{\infty} \|f_k\|_{L_p} \leqslant 2^{-(n+1)\gamma} \sum_{k=n+1}^{\infty} 2^{k\gamma} \|f_k\|_{L_p}$$

$$\leqslant h^{\gamma} \|f\|_{B_p^{\gamma, 1}(\eta)}. \quad \blacksquare$$

6. THE PROOFS OF THE MAIN RESULTS

In this section, we prove Theorem 2.3, Proposition 2.8, Theorem 2.10, and Theorem 2.11. The technique used in proving Theorem 2.3 might well be called *approximation by replacement*. In order to approximate $f \in L_p$ from $S_p^h(\phi_h)$, we start with a very good approximation to f written using various dilates of the shifts of a certain function η (i.e., the scheme described in Lemma 5.12). By *replacing* each instance of η with an approximation to η from an appropriate dilate of $S_1(\phi_h)$, we then obtain an approximation to f from $S_p^h(\phi_h)$ whose closeness to f can be estimated in terms of how well each replacement actually approximates η. It turns out that these replacements need to approximate η not in L_p, but rather in $\mathscr{L}_p$. This is because η appears in expressions like $\eta *' a$, $a \in l_p$, where the smallness of $\|\psi - \eta\|_{L_p}$ does not ensure the smallness (relative to $\|a\|_{l_p}$) of $\|(\psi - \eta) *' a\|_{L_p}$, whereas the smallness of $\|\psi - \eta\|_{\mathscr{L}_p}$ does. Ultimately, the $\mathscr{L}_p$-distance between η and various dilates of $S_1(\phi_h)$ becomes the issue as reflected in the hypothesis of the theorem.

Proof of Theorem 2.3. By (2.4) there exists $A \in (0..0)$ such that

$$\sup_{0<r\leqslant h} \operatorname{dist}(\eta, S_1^h(\phi_r); \mathscr{L}_{\bar{p}}) < Ah^{\gamma} \qquad \forall h \in (0..h_0]. \tag{6.1}$$

Let $1 \leqslant p \leqslant \bar{p}$. Since $\|\cdot\|_{\mathscr{L}_p} \leqslant \|\cdot\|_{\mathscr{L}_{\bar{p}}}$, we may assume WLOG that $\bar{p}=p$. Also, there is no loss of generality in assuming that $h_0=1$. Let $h \in (0\,..\,1]$ and let $n := n(h)$ be the largest integer for which $h2^n \leqslant 1$. Let $\gamma > 0$ and let $f \in B_p^{\gamma,1}$. Let $\{f_k\}_{k \in \mathbb{Z}_+}$ be as in (2.1). We proceed now to define our approximation to f from $S_p^h(\phi_h)$. By (6.1), there exist $g_k \in S_1(\phi_h)$ such that

$$\|\eta - g_k(2^{n-k}\cdot)\|_{\mathscr{L}_p} \leqslant A\,2^{-\gamma(n-k)}, \qquad 0 \leqslant k \leqslant n. \tag{6.2}$$

(Note: $2^{-(n-k)}$ is playing the role of h in (6.1), while h is playing the role of r in (6.1). Equation (6.2) is a valid application of (6.1) because $0 < h \leqslant 2^{-(n-k)} \leqslant 1$.) Since $g_k \in S_1(\phi_h)$, it follows from the fact that $\mathscr{L}_p$ is a Banach space that $g_k \in \mathscr{L}_p$. Note that, by Lemma 5.12, $\|f_k\|_{l_p(h2^{n-k}\mathbb{Z}^d)} < \infty$ and hence

$$g_k(2^{n-k}\cdot) *'_{h2^{n-k}} f_k = \sum_{j \in \mathbb{Z}^d} f_k(h2^{n-k}j)\, g_k(\cdot/h - 2^{n-k}j) \in S_p^h(g_k), \qquad 0 \leqslant k \leqslant n.$$

Since $g_k \in S_1(\phi_h)$, it follows that $S_p(g_k) \subseteq S_p(\phi_h)$. Hence,

$$s_h := \sum_{k=0}^{n} g_k(2^{n-k}\cdot) *'_{h2^{n-k}} f_k \in S_p^h(\phi_h).$$

Now,

$$\left\| \sum_{k=0}^{n} f_k - s_h \right\|_{L_p} = \left\| \sum_{k=0}^{n} (\eta - g_k(2^{n-k}\cdot)) *'_{h2^{n-k}} f_k \right\|_{L_p},$$

by Lemma 5.12 (1),

$$\leqslant \sum_{k=0}^{n} \|\eta - g_k(2^{n-k}\cdot)\|_{\mathscr{L}_p} (h2^{n-k})^{d/p} \|f_k\|_{l_p(h2^{n-k}\mathbb{Z}^d)},$$

by Lemma 5.1,

$$\leqslant \sum_{k=0}^{n} A\,2^{-\gamma(n-k)} \operatorname{const}(d)\, \|f_k\|_{L_p},$$

by (6.2) and Lemma 5.12 (2),

$$\leqslant \operatorname{const}(d)\, A\,2^{-\gamma n} \sum_{k=0}^{\infty} 2^{k\gamma} \|f_k\|_{L_p}$$

$$\leqslant \operatorname{const}(d, \gamma)\, Ah^{\gamma}\, \|f\|_{B_p^{\gamma,1}(1)}. \tag{6.3}$$

Therefore, by (6.3) and Lemma 5.12 (3), we conclude that

$$\operatorname{dist}(f, S_p^h(\phi_h); L_p) \leqslant \|f - s_h\|_{L_p} \leqslant \left\| f - \sum_{k=0}^{n} f_k \right\|_{L_p} + \left\| \sum_{k=0}^{n} f_k - s_h \right\|_{L_p}$$

$$\leqslant \operatorname{const}(d, A, \gamma)\, h^{\gamma}\, \|f\|_{B_p^{\gamma,1}(\eta)}. \quad ∎$$

Proof of Proposition 2.8. Let $0 < r \leqslant h \leqslant h_0$. Put $\hat{f} := \hat{\eta}/\hat{\phi}_r(h\cdot)$. Then by Wiener's lemma (Lemma 5.11), $f \in L_1$. Since $\hat{f}$ is compactly supported, we have by Lemma 5.10 that $\|f\|_{l_1(h\mathbb{Z}^d)} < \infty$. Hence $\phi_r *'_h f \in S_1^h(\phi_r)$ and

$$(\phi_r *'_h f)^{\wedge} = \hat{\phi}_r(h\cdot) \sum_{j \in \mathbb{Z}^d} \frac{\hat{\eta}(\cdot - 2\pi j/h)}{\hat{\phi}_r(h\cdot - 2\pi j)}, \qquad \text{by Lemma 5.10,}$$

$$= \hat{\eta} + \hat{\phi}_r(h\cdot) \sum_{j \in \mathbb{Z}^d \backslash 0} \frac{\hat{\eta}(\cdot - 2\pi j/h)}{\hat{\phi}_r(h\cdot - 2\pi j)}.$$

Thus,

$$\operatorname{dist}(\eta, S_1^h(\phi_r); \mathscr{L}_{\bar{p}}) \leqslant \|\phi_r *'_h f - \eta\|_{\mathscr{L}_{\bar{p}}} = \left\| \left(\hat{\phi}_r(h\cdot) \sum_{j \in \mathbb{Z}^d \backslash 0} \frac{\hat{\eta}(\cdot + 2\pi j/h)}{\hat{\phi}_r(h\cdot + 2\pi j)} \right)^{\wedge} \right\|_{\mathscr{L}_{\bar{p}}}.$$

Hence (1). For the sake of proving (2), we may assume WLOG that

$$\sum_{j \in \mathbb{Z}^d \backslash 0} \left\| \left(\frac{\hat{\eta}\hat{\phi}_r(h\cdot + 2\pi j)}{\hat{\phi}_r(h\cdot)} \right)^{\vee} \right\|_{\mathscr{L}_{\bar{p}}} < \infty. \tag{6.4}$$

Hence,

$$\sum_{j \in \mathbb{Z}^d \backslash 0} \left\| \left(\frac{\hat{\eta}\hat{\phi}_r(h\cdot + 2\pi j)}{\hat{\phi}_r(h\cdot)} \right)^{\vee} \right\|_{\mathscr{L}_{\bar{p}}} = \sum_{j \in \mathbb{Z}^d \backslash 0} \left\| \left(\hat{\phi}_r(h\cdot) \frac{\hat{\eta}(\cdot - 2\pi j/h)}{\hat{\phi}_r(h\cdot - 2\pi j)} \right)^{\vee} \right\|_{\mathscr{L}_{\bar{p}}}$$

$$\geqslant \left\| \sum_{j \in \mathbb{Z}^d \backslash 0} \left(\hat{\phi}_r(h\cdot) \frac{\hat{\eta}(\cdot - 2\pi j/h)}{\hat{\phi}_r(h\cdot - 2\pi j)} \right)^{\vee} \right\|_{\mathscr{L}_{\bar{p}}}$$

$$= \left\| \left(\hat{\phi}_r(h\cdot) \sum_{j \in \mathbb{Z}^d \backslash 0} \frac{\hat{\eta}(\cdot + 2\pi j/h)}{\hat{\phi}_r(h\cdot + 2\pi j)} \right)^{\vee} \right\|_{\mathscr{L}_{\bar{p}}},$$

where the first inequality and the last equality follow from the finiteness of the second expression which follows from (6.4) and the first equality. Thus (2) follows from (1).

We consider now (3) where it is assumed that $2 \leqslant \bar{p} \leqslant \infty$. By Lemma 5.2,

$$\left\| \left(\hat{\phi}_r(h\cdot) \sum_{j \in \mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\cdot + 2\pi j/h)}{\hat{\phi}_r(h\cdot + 2\pi j)} \right)^{\vee} \right\|_{\mathscr{L}_{\bar{p}}}$$

$$\leqslant \text{const}(d, \bar{p}) \left\| \hat{\phi}_r(h\cdot) \sum_{j \in \mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\cdot + 2\pi j/h)}{\hat{\phi}_r(h\cdot + 2\pi j)} \right\|_{W^m_{\bar{q}}(\mathbb{R}^d)}$$

$$= \text{const}(d, \bar{p}) \left(\sum_{j \in \mathbb{Z}^d \setminus 0} \left\| \hat{\phi}_r(h\cdot) \frac{\hat{\eta}(\cdot + 2\pi j/h)}{\hat{\phi}_r(h\cdot + 2\pi j)} \right\|^{\bar{q}}_{W^m_{\bar{q}}(\delta C + 2\pi j/h)} \right)^{1/\bar{q}},$$

$$\text{since} \quad \operatorname{supp} \hat{\eta} \subset \delta C,$$

$$= \text{const}(d, \bar{p}, \eta) \left(\sum_{j \in \mathbb{Z}^d \setminus 0} \left\| \frac{\hat{\phi}_r(h\cdot + 2\pi j)}{\hat{\phi}_r(h\cdot)} \right\|^{\bar{q}}_{W^m_{\bar{q}}(\delta C)} \right)^{1/\bar{q}}, \qquad \text{since} \quad \hat{\eta} \in C^\infty_c.$$

Hence (3) follows from (1). ∎

Proof of Theorem 2.10. First note that $S_1(\psi_h) \subseteq S_1(\phi_h)$ because $\psi_h \in S_1(\phi_h)$. Hence, $\psi_r *'_h \eta \in S^h_1(\phi_r)$, $0 < r \leqslant h \leqslant h_0$. Let $\sigma \in C^\infty_c$ be such that

$$\sum_{j \in \mathbb{Z}^d} \sigma(\cdot + j) = 1.$$

Fix $0 < r \leqslant h \leqslant h_0$. Then

$$\operatorname{dist}(\eta, S^h_1(\phi_r); \mathscr{L}_{\bar{p}}) \leqslant \|\eta - \psi_r *'_h \eta\|_{\mathscr{L}_{\bar{p}}}$$

$$= \left\| \sum_{j \in \mathbb{Z}^d} \sigma(\cdot + j)\, \eta - \psi_r *'_h \left(\sum_{j \in \mathbb{Z}^d} \sigma(\cdot + j)\, \eta \right) \right\|_{\mathscr{L}_{\bar{p}}},$$

$$\text{since} \quad \eta = \sum_{j \in \mathbb{Z}^d} \sigma(\cdot + j)\, \eta,$$

$$= \left\| \sum_{j \in \mathbb{Z}^d} (\sigma(\cdot + j)\, \eta - \psi_r *'_h (\sigma(\cdot + j)\, \eta)) \right\|_{\mathscr{L}_{\bar{p}}}$$

$$\leqslant \sum_{j \in \mathbb{Z}^d} \|\sigma(\cdot + j)\, \eta - \psi *'_h (\sigma(\cdot + j)\, \eta)\|_{\mathscr{L}_{\bar{p}}}$$

$$\leqslant \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \|\sigma(\cdot + j)\, \eta - \psi_r *'_h (\sigma(\cdot + j)\, \eta)\|_{L_{\bar{p}}(k - j + C)}. \tag{6.5}$$

Let $m > 2$ be so large that $\operatorname{supp}\sigma \subset mC$ and $\operatorname{supp}\sigma \cap (k + C) = \varnothing$ whenever $|k| \geqslant md$. Now,

$$\begin{aligned}
&\sum_{j \in \mathbb{Z}^d} \sum_{|k| < md} \|\sigma(\cdot + j)\,\eta - \psi_r *'_h (\sigma(\cdot + j)\,\eta)\|_{L_{\bar{p}}(k - j + C)} \\
&\quad \leqslant \sum_{j \in \mathbb{Z}^d} \#\{|k| < md\}\ \|\sigma(\cdot + j)\,\eta - \psi_r *'_h (\sigma(\cdot + j)\,\eta)\|_{L_{\bar{p}}} \\
&\quad \leqslant \sum_{j \in \mathbb{Z}^d} (2md)^d\, c_2\, \|\sigma(\cdot + j)\,\eta\|_N\, h^{\gamma} \\
&\quad = \operatorname{const}(d, \sigma, \eta, c_2, N)\, h^{\gamma}, \qquad \text{by (ii)}.
\end{aligned} \tag{6.6}$$

And,

$$\begin{aligned}
&\sum_{j \in \mathbb{Z}^d} \sum_{|k| \geqslant md} \|\psi_r *'_h (\sigma(\cdot + j)\,\eta)\|_{L_{\bar{p}}(k - j + C)} \\
&\quad \leqslant \sum_{j \in \mathbb{Z}^d} \sum_{|k| \geqslant md} \sum_{l \in \mathbb{Z}^d} \|\sigma(hl + j)\,\eta(hl)\,\psi_r(\cdot/h - l)\|_{L_\infty(k - j + C)} \\
&\quad \leqslant \sum_{j \in \mathbb{Z}^d} \sum_{|k| \geqslant md} \operatorname{const}(d, \sigma)\, h^{-d}\, \|\sigma(h\cdot + j)\,\eta(h\cdot)\|_{L_\infty} \\
&\qquad \times \|\psi_r\|_{L_\infty(h^{-1}(k + (m+1)\,C))}, \\
&\qquad\qquad \text{since} \quad \sigma(hl + j) \neq 0 \qquad \text{only if} \quad l \in h^{-1}(mC - j), \\
&\quad \leqslant \sum_{|k| \geqslant md} \operatorname{const}(d, \sigma, \gamma)\, h^{-d}\, c_1 (1 + |k|/h)^{-(d+\gamma)} \sum_{j \in \mathbb{Z}^d} \|\sigma(\cdot + j)\,\eta\|_{L_\infty}, \\
&\qquad\qquad \text{by (i)}, \\
&\quad \leqslant \operatorname{const}(d, \sigma, \eta, c_1, \gamma)\, h^{\gamma}.
\end{aligned} \tag{6.7}$$

Therefore, by (6.5), (6.6), and (6.7),

$$\operatorname{dist}(\eta, S_1^h(\phi_r); \mathscr{L}_{\bar{p}}) \leqslant \operatorname{const}(d, \sigma, \eta, c_1, c_2, \gamma, N)\, h^{\gamma}. \quad \blacksquare$$

We make use of the following lemma in the proof of Theorem 2.11.

LEMMA 6.8. *Let* $\phi_h \in L_\infty$, $h \in (0\,..\,h_0]$. *Assume that there exists* $\gamma \in (0\,..\,\infty)$ *such that for some* $c, \varepsilon \in (0\,..\,\infty)$,

(i) $|\phi_h(x)| \leqslant c(1 + |x|)^{-(d + \lceil \gamma \rceil + \varepsilon)}, \qquad \forall x \in \mathbb{R}^d, \qquad h \in (0\,..\,h_0];$

(ii) $m := \inf_{h \in (0\,..\,h_0]} |\hat{\phi}_h(0)| > 0.$

Put $\mathcal{N} := \{ j \in \mathbb{Z}_+^d : |j|_1 < \gamma \}$. *Then there exists* $c_h \in l_1(\mathbb{Z}^d)$ *with* $\operatorname{supp} c_h \subseteq \mathcal{N}$ *such that with* $\psi_h := \phi_h *' c_h$,

(1) $\|c_h\|_{l_1} \leqslant \mathrm{const}(d, \gamma, c, \varepsilon, m) \qquad \forall h \in (0..h_0]$;

(2) $(D^\alpha \hat{\psi}_h)(0) = \delta_{0\alpha} \qquad \forall |\alpha| < \gamma, \qquad h \in (0..h_0]$.

Proof. Put $k := \lceil \gamma \rceil$ and for functions f which are C^{k-1} in a neighbourhood of 0, let $P_{k-1} f$ be the unique polynomial of total degree $<k$ which satisfies $(D^\alpha P_{k-1} f)(0) = (D^\alpha f)(0)$ for all $|\alpha| < k$. In other words,

$$P_{k-1} f := \sum_{|\alpha| < k} \frac{(D^\alpha f)(0)}{\alpha!} (\ )^\alpha.$$

Fix $h \in (0..h_0]$. It follows from (i) that $\hat{\phi}_h \in C^k(\mathbb{R}^d)$. Put $p_h := P_{k-1} \hat{\phi}_h$ and $g_h := P_{k-1}(1/\hat{\phi}_h)$, say $p_h = \sum_{|\alpha| < k} a_\alpha (\ )^\alpha$ and $g_h = \sum_{|\alpha| < k} b_\alpha (\ )^\alpha$. Note that

$$P_{k-1}(p_h g_h) = P_{k-1} \left(\frac{\hat{\phi}_h}{\hat{\phi}_h} \right) = 1.$$

Hence, $\sum_{\beta \leqslant \alpha} b_\beta a_{\alpha - \beta} = \delta_{0\alpha}$, which allows the b_α's to be solved recursively by

$$b_\alpha = a_0^{-1} \left(\delta_{0\alpha} - \sum_{\beta < \alpha} b_\beta a_{\alpha - \beta} \right). \tag{6.9}$$

Now, $a_0^{-1} = \hat{\phi}_h(0)^{-1} \leqslant m^{-1}$ by (ii), and it follows from (i) that

$$|a_\alpha| \leqslant \mathrm{const}(d, \gamma, c, \varepsilon) \qquad \forall\ |\alpha| < k.$$

Hence, by (6.9),

$$|b_\alpha| \leqslant \mathrm{const}(d, \gamma, c, \varepsilon, m) \qquad \forall\ |\alpha| < k.$$

Claim 6.11. The mapping

$$q \mapsto P_{k-1} \sum_{j \in \mathcal{N}} q(j)\, e_{-j}$$

is a linear bijection of $\mathbb{C}^{\mathcal{N}}$ onto Π_{k-1}. In particular, it is invertible.

Proof. It was shown in [9, Corollary 3.36] that the mapping

$$q \mapsto P_{k-1} \sum_{j \in \mathcal{N}} q(j)\, e_{-ij}$$

is a linear bijection of $\mathbb{C}^{\mathcal{N}}$ onto Π_{k-1}. So for each $\alpha \in \mathcal{N}$, there exists $q_\alpha \in \mathbb{C}^{\mathcal{N}}$ such that $P_{k-1} \sum_{j \in \mathcal{N}} q_\alpha(j)\, e_{-ij} = (\ )^\alpha$. Hence,

$$\sum_{j \in \mathcal{N}} q_\alpha(j) \frac{(j \cdot)^n}{n!} = \begin{cases} 0, & \text{if } \ 0 \leqslant n < k, \quad n \neq |\alpha|; \\ (\ )^\alpha, & \text{if } \ n = |\alpha|. \end{cases}$$

Therefore,

$$\sum_{j \in \mathcal{N}} q_\alpha(j) \frac{(-ij \cdot)^n}{n!} = \begin{cases} 0, & \text{if } \ 0 \leqslant n < k, \quad n \neq |\alpha|; \\ (-i)^n (\ )^\alpha, & \text{if } \ n = |\alpha|. \end{cases}$$

Or, in other words, $P_{k-1} \sum_{j \in \mathcal{N}} q(j)\, e_{-j} = (-i)^{|\alpha|} (\ )^\alpha$ for all $\alpha \in \mathcal{N}$. Since $\{(\ )^\alpha : \alpha \in \mathcal{N}\}$ is a basis of Π_{k-1} (and since $\dim \mathbb{C}^{\mathcal{N}} = \dim \Pi_{k-1}$), the claim is proved.

As a consequence of Claim 6.11, it follows that there exists $c_h \in l_1(\mathbb{Z}^d)$ with $\operatorname{supp} c_h \subseteq \mathcal{N}$ such that

$$g_h = P_{k-1} \sum_{\alpha \in \mathcal{N}} c_h(\alpha)\, e_{-\alpha},$$

and

$$\|c_h\|_{l_1} \leqslant \text{const}(d, \gamma) \max_{|\alpha| < k} |b_\alpha|.$$

Thus, by (6.10), (1) is established. Put $\psi_h := \phi *' c_h$. Then, since $(\phi_h(\cdot - \alpha))\,\hat{} = \hat{\phi}_h e_{-\alpha}$, it follows that $\hat{\psi}_h = \hat{\phi}_h \sum_{\alpha \in \mathcal{N}} c_h(\alpha)\, e_{-\alpha}$. Hence

$$P_{k-1} \hat{\psi}_h = P_{k-1}(\hat{\phi}_h g_h) = P_{k-1}(p_h g_h) = 1.$$

Therefore, (2) holds. ∎

Proof of Theorem 2.11. We will be employing Theorem 2.10. Put $k := \lceil \gamma \rceil$. Let $\mathcal{N}$, m, c_h, and ψ_h be as in Lemma 6.8. Since $\operatorname{supp} c_h \subseteq \mathcal{N}$ and in view of Lemma 6.8 (1), it follows that

$$|\psi_h(x)| \leqslant \text{const}(d, \gamma, c, \varepsilon, m)(1 + |x|)^{-(d+k+\varepsilon)}, \qquad \forall x \in \mathbb{R}^d, \qquad h \in (0 .. h_0]. \tag{6.12}$$

In particular, condition(i) of Theorem 2.10 is satisfied. We now turn toward the task of showing that condition(ii) of Theorem 2.10 holds. Let N be the least positive integer for which

$$\sup_{f \in C_c^\infty} \frac{\| |\cdot|^{2k} \hat{f} \|_{W_1^{d+1}(\mathbb{R}^d)} + \| (1 + |\cdot|^k) \hat{f} \|_{L_{\bar{q}}}}{\|f\|_N} < \infty.$$

Let $f \in C_c^\infty$, $h \in (0..h_0]$ and $r \in (0..h]$. Define f_h by $\hat{f}_h := \hat{\eta}(h\cdot)\,\hat{f}$. Note that

$$\|\psi_r *'_h f - f\|_{L_{\bar{p}}} \leqslant \|\psi_r *'_h (f - f_h)\|_{L_{\bar{p}}} + \|f_h - f\|_{L_{\bar{p}}} + \|\psi_r *'_h f_h - f_h\|_{L_{\bar{p}}}. \quad (6.13)$$

Claim 6.14.

$$\|\psi_r *'_h (f - f_h)\|_{L_{\bar{p}}} + \|f_h - f\|_{L_{\bar{p}}} \leqslant \text{const}(d, \gamma, c, \varepsilon, m, \delta)\, h^\gamma\, \|f\|_N.$$

Proof. By Lemma 5.1,

$$\begin{aligned}
\|\psi_r *'_h (f - f_h)\|_{L_{\bar{p}}} &\leqslant \|\psi_r\|_{\mathscr{L}_{\bar{p}}}\, h^{d/\bar{p}}\, \|f - f_h\|_{l_{\bar{p}}(h\mathbb{Z}^d)} \\
&\leqslant \text{const}(d, \gamma, c, \varepsilon, m)\, h^{d/\bar{p}} \|f - f_h\|_{l_{\bar{p}}(h\mathbb{Z}^d)}, \qquad \text{by (6.12)} \\
&\leqslant \text{const}(d, \gamma, c, \varepsilon, m)\, \|(1 + |x|)^{d+1}\, (f - f_h)\|_{L_\infty}.
\end{aligned}$$

Since it is also true that $\|f_h - f\|_{L_{\bar{p}}} \leqslant \text{const}(d)\, \|(1 + |x|)^{d+1}\, (f - f_h)\|_{L_\infty}$, in order to prove the claim, it suffices to show that

$$\|(1 + |x|)^{d+1}\, (f - f_h)\|_{L_\infty} \leqslant \text{const}(d, \gamma, \delta)\, h^\gamma\, \|f\|_N.$$

Now

$$\begin{aligned}
&\|(1 + |x|)^{d+1}\, (f - f_h)\|_{L_\infty} \\
&\quad \leqslant \text{const}(d)\, \|(1 - \hat{\eta}(h\cdot))\, \hat{f}\|_{W_1^{d+1}(\mathbb{R}^d)} \\
&\quad \leqslant \text{const}(d, \gamma)\, h^{2k}\, \|(1 - \hat{\eta}(h\cdot))\, |h\cdot|^{-2k}\|_{W_\infty^{d+1}(\mathbb{R}^d)}\, \||\cdot|^{2k}\, \hat{f}\|_{W_1^{d+1}(\mathbb{R}^d)} \\
&\quad \leqslant \text{const}(d, \gamma, \delta)\, h^\gamma\, \||\cdot|^{2k}\, \hat{f}\|_{W_1^{d+1}(\mathbb{R}^d)}, \qquad \text{since} \quad \hat{\eta} = 1 \quad \text{on } \tfrac{1}{2}\,\delta C, \\
&\quad \leqslant \text{const}(d, \gamma, \delta)\, h^\gamma\, \|f\|_N.
\end{aligned}$$

Hence the claim.

In view of (6.13) and Claim 6.14, condition (ii) of Theorem 2.10 will be satisfied if we show that there exists $N \in \mathbb{N}$ such that

$$\|\psi_r *'_h f_h - f_h\|_{L_{\bar{p}}} \leqslant \text{const}(d, \gamma, c, \varepsilon, m, \delta)\, h^\gamma\, \|f\|_N. \quad (6.15)$$

By the Hausdorff–Young theorem (cf. 23, p. 142])

$$\begin{aligned}
&\|\psi_r *'_h f_h - f_h\|_{L_{\bar{p}}} \\
&\quad \leqslant \text{const}(d)\, \|(\psi_r *'_h f_h - f_h)^\wedge\|_{L_{\bar{q}}} \\
&\quad = \text{const}(d) \left\| \hat{\psi}_r(h\cdot) \sum_{j \in \mathbb{Z}^d} \hat{f}_h(\cdot - 2\pi j/h) - \hat{f}_h \right\|_{L_{\bar{q}}}, \qquad \text{by Lemma 5.10,} \\
&\quad \leqslant \text{const}(d)\, \|(\hat{\psi}_r(h\cdot) - 1)\, \hat{f}_h\|_{L_{\bar{q}}} \\
&\qquad + \text{const}(d) \left(\sum_{j \in \mathbb{Z}^d \backslash 0} \|\hat{\psi}_r(h\cdot + 2\pi j)\, \hat{f}_h\|_{L_{\bar{q}}}^{\bar{q}} \right)^{1/\bar{q}}, \qquad (6.16)
\end{aligned}$$

since $\operatorname{supp} \hat{f}_h \subseteq h^{-1} \delta C$. By Lemma 6.8 (2),

$$|\hat{\psi}_r(x) - 1| \leqslant \operatorname{const}(d, \gamma)\, |x|^k\, \|\hat{\psi}_r\|_{W^k_\infty(\delta C)} \qquad \forall x \in \delta C.$$

By (6.12), $\|\hat{\psi}_r\|_{W^k_\infty(\delta C)} \leqslant \operatorname{const}(d, \gamma, c, \varepsilon, m)$. Hence

$$\left\| \frac{\hat{\psi}_r - 1}{|\cdot|^k} \right\|_{L_\infty(\delta C)} \leqslant \operatorname{const}(d, \gamma, c, \varepsilon, m).$$

Therefore,

$$\begin{aligned}
\|(\hat{\psi}_r(h\cdot) - 1)\, \hat{f}_h\|_{L_{\bar{q}}} &= h^k \left\| \frac{\hat{\psi}_r(h\cdot) - 1}{|h\cdot|^k}\, |\cdot|^k\, \hat{\eta}(h\cdot)\, \hat{f} \right\|_{L_{\bar{q}}} \\
&\leqslant h^k \left\| \frac{\hat{\psi}_r - 1}{|\cdot|^k} \right\|_{L_\infty(\delta C)} \||\cdot|^k \hat{f}\|_{L_{\bar{q}}}, \qquad \text{since} \quad \operatorname{supp} \hat{\eta} \subset \delta C, \\
&\leqslant \operatorname{const}(d, \gamma, c, \varepsilon, m)\, h^k\, \|f\|_N. \qquad (6.17)
\end{aligned}$$

Now, for $j \in \mathbb{Z}^d \backslash 0$,

$$\begin{aligned}
\|\hat{\psi}_r(h\cdot + 2\pi j)\, \hat{f}_h\|_{L_{\bar{q}}} &= h^\gamma \left\| \frac{\hat{\psi}_r(h\cdot + 2\pi j)}{h^\gamma + |h\cdot|^\gamma} (1 + |\cdot|^\gamma)\, \hat{\eta}(h\cdot)\, \hat{f} \right\|_{L_{\bar{q}}(h^{-1}\delta C)} \\
&\leqslant h^\gamma \left\| \frac{\hat{\psi}_r(\cdot + 2\pi j)}{h^\gamma + |\cdot|^\gamma} \right\|_{L_\infty(\delta C)} \|(1 + |\cdot|^\gamma)\, \hat{f}\|_{L_{\bar{q}}} \\
&\leqslant \operatorname{const}(d, \gamma)\, h^\gamma \left\| \frac{\hat{\psi}_r(\cdot + 2\pi j)}{r^\gamma + |\cdot|^\gamma} \right\|_{L_\infty(\delta C)} \|f\|_N, \\
&\qquad \text{since} \quad 0 < r \leqslant h, \\
&\leqslant \operatorname{const}(d, \gamma, c, \varepsilon, m)\, h^\gamma \left\| \frac{\hat{\phi}_r(\cdot + 2\pi j)}{r^\gamma + |\cdot|^\gamma} \right\|_{L_\infty(\delta C)} \|f\|_N,
\end{aligned}$$

since $\hat{\psi}_r = \hat{\phi}_r \sum_{\alpha \in \mathcal{N}} c_r(\alpha)\, e_{-\alpha}$ and $\|\sum_{\alpha \in \mathcal{N}} c_r(\alpha)\, e_{-\alpha}\|_{L_\infty} \leqslant \|c_r\|_{l_1} \leqslant \operatorname{const}(d, \gamma, c, m)$ by Lemma 6.8 (1). Therefore

$$\begin{aligned}
&\left(\sum_{j \in \mathbb{Z}^d \backslash 0} \|\hat{\psi}_r(h\cdot + 2\pi j)\, \hat{f}_h\|^{\bar{q}}_{L_{\bar{q}}} \right)^{1/\bar{q}} \\
&\qquad \leqslant \operatorname{const}(d, \gamma, c, \varepsilon, m)\, h^\gamma\, \|f\|_N\, A(\delta, \gamma, \bar{q}). \qquad (6.18)
\end{aligned}$$

Hence, by (6.15), (6.16), (6.17), and (6.18), we conclude that condition (ii) of Theorem (2.10) is satisfied. The proof is now completed by applying Theorem 2.10. ∎

7. THE STRANG–FIX CONDITIONS

In this section we address the task of finding reasonable side conditions under which it can be proven that if ϕ satisfies the Strang–Fix conditions of order k and $\hat{\phi}(0) \neq 0$, then $(S^h(\phi))_h$ provides L_p-approximation of order k. For example, in Theorem 1.4 (by Jia and Lei) it is proven that under conditions (i) and (ii) of Theorem 1.4, the Strang–Fix conditions guarantee "controlled" L_p-approximation of order k for all $1 \leqslant p \leqslant \infty$. The problem with the strong decay assumption of condition (i) is that it implies that $\hat{\phi}$ is globally smooth, whereas in some applications, $\hat{\phi}$ is only smooth away from the origin. It is thus desirable to find side conditions which do not require $\hat{\phi}$ to be smooth near the origin. This was achieved for $p=2$ by de Boor *et al.* in [4]. In order to state their result we introduce the potential spaces

$$W_2^\rho := \{ f \in L_2 \colon \| f \|_{W_2^\rho} := \| (1 + |\cdot|^2)^{\rho/2} \hat{f} \|_{L_2} < \infty \}, \qquad \rho \geqslant 0.$$

We also need local versions of these spaces. If ρ is an integer and $\Omega \subset \mathbb{R}^d$ is open, then $W_2^\rho(\Omega)$ is simply the Sobolev space defined in Section 1. It is fairly easy to see by the Plancherel theorem (cf. [32, Theorem 7.9]) that $W_2^\rho(\mathbb{R}^d) = W_2^\rho$ and that their norms are equivalent. In this case ($\rho \in \mathbb{Z}_+$), if $\{\Omega_\beta\}_\beta$ is a collection of disjoint open sets, then with $\Omega := \bigcup_\beta \Omega_\beta$

$$\sum_\beta \| f \|^2_{W_2^\rho(\Omega_\beta)} = \| f \|^2_{W_2^\rho(\Omega)}.$$

For non-integer ρ, there are several equivalent ways of defining $W_2^\rho(\Omega)$ (cf. [1, Chap. 7]) so that $W_2^\rho(\mathbb{R}^d) = W_2^\rho$ (with equivalent norms). In this case we have the subadditive property

$$\sum_\beta \| f \|^2_{W_2^\rho(\Omega_\beta)} \leqslant \mathrm{const}(d, \rho, \{\Omega_\beta\}_\beta) \, \| f \|^2_{W_2^\rho(\Omega)}, \tag{7.1}$$

whenever, say, $\{\Omega_\beta\}_\beta$ is a disjoint collection of cubes and $\Omega := \bigcup_\beta \Omega_\beta$. We can now state the relevent result of [4].

THEOREM 7.2. *Let $\phi \in L_2$ and $k \in \mathbb{N}$. Assume that $\hat{\phi} \in W_2^\rho(\delta C + 2\pi \mathbb{Z}^d \backslash 0)$ and $\hat{\phi} > \varepsilon$ a.e. on δC for some $\delta, \varepsilon > 0$ and $\rho > k + d/2$. If ϕ satisfies the Strang–Fix conditions of order k, then the stationary ladder $(S^h(\phi))_h$ provides L_2-approximation of order k.*

Proof. cf. [4; Theorem 5.14].

Note that the side condition, $\hat{\phi} \in W_2^\rho(\delta C + 2\pi \mathbb{Z}^d \backslash 0)$, does not impose any smoothness on $\hat{\phi}$ near the origin and is implied by a strong decay of ϕ (e.g., condition (i) of Theorem 1.4).

There have been other attempts to give side conditions under which the Strang–Fix conditions of order k and $\hat{\phi}(0) \neq 0$ imply L_p-approximation of order k (say, in the sense of (1.1)), namely, [8, Theorem 3.5] ($p = \infty$) and [21, Theorem 3.9 ($2 \leqslant p < \infty$) and Theorem 4.8 ($1 < p < \infty$)]. When $2 < p \leqslant \infty$, the above-mentioned results are successful in that their side conditions require no smoothness of $\hat{\phi}$ near the origin, but fall short of the standard established by [4] in that their side conditions are not implied by a strong decay of ϕ. The side conditions of [21, Theorem 4.8] require a smoothness (increasing with k) of $\hat{\phi}$ near the origin and are not implied by a strong decay of ϕ.

We state now the present contributions which derive from Theorem 2.3 in conjunction with Proposition 2.8 (1).

Theorem 7.3. *Let* $\bar{p} \in \{1, 2\}$. *Let* $\phi \in \mathscr{L}_{\bar{p}}$ *and* $k \in \mathbb{N}$ *be such that* $\hat{\phi} \in W_2^{\rho}(\varepsilon C + 2\pi\mathbb{Z}^d \backslash 0)$ *for some* $\varepsilon \in (0 \,..\, 2\pi)$ *and* $\rho > k + d/2$. *In case* $\bar{p} = 2$ *assume additionally that* $\hat{\phi} \in C^m(\varepsilon C)$, *where* m *is the least integer satisfying* $m > d/2$. *If* $\hat{\phi}(0) \neq 0$ *and* ϕ *satisfies the Strang–Fix conditions of order* k (1.3), *then the stationary ladder* $(S_p^h(\phi))_h$ *provides* L_p*-approximation of order* k *for all* $1 \leqslant p \leqslant \bar{p}$.

Note that the case $\bar{p} = 1$ is very satisfactory in that the side conditions impose no smoothness assumption on $\hat{\phi}$ near the origin and they are implied by a strong decay of ϕ (e.g., condition (i) of Theorem 1.4). However, for the case $\bar{p} = 2$, we do impose a (fixed) smoothness assumption on $\hat{\phi}$ near the origin. Nonetheless, the side conditions are implied by a sufficiently strong decay of ϕ (e.g., if $k > d/2$, then condition (i) of Theorem 1.4 suffices).

Our result for the case $2 < \bar{p} \leqslant \infty$ is as follows.

Theorem 7.4. *Let* $2 < \bar{p} \leqslant \infty$ *and let* $\bar{q}$ *be the exponent conjugate to* $\bar{p}$ (*i.e., satisfying* $1/\bar{p} + 1/\bar{q} = 1$). *Let* m *be the least integer greater than* $d/\bar{q}$. *Let* $k \in \mathbb{N}$ *and define*

$$\rho := \begin{cases} k + d, & \text{if } \ \bar{p} = \infty; \\ \min \mathbb{N} \cap (k + d/\bar{q} \,..\, \infty), & \text{if } \ 2 < \bar{p} < \infty. \end{cases}$$

Let $\phi \in \mathscr{L}_{\bar{p}}$ *satisfy* $\hat{\phi} \in C^m(\varepsilon C)$ *and* $\hat{\phi} \in W_{\bar{q}}^{\rho}(\varepsilon C + 2\pi\mathbb{Z}^d \backslash 0)$ *for some* $\varepsilon > 0$. *If* $\hat{\phi}(0) \neq 0$ *and* ϕ *satisfies the Strang–Fix conditions of order* k (1.3), *then the stationary ladder* $(S_p^h(\phi))_h$ *provides* L_p*-approximation of order* k *for all* $1 \leqslant p \leqslant \bar{p}$.

Note that the side conditions impose a (fixed) smoothness assumption of $\hat{\phi}$ near the origin, and they are not implied by a strong decay of ϕ.

Proof of Theorem 7.3 *and Theorem* 7.4. In case $\bar{p} \in \{1, 2\}$, put $\bar{q} := 2$. Assume that ϕ satisfies the Strang–Fix conditions of order k and $\hat{\phi}(0) \neq 0$. Then there exists $\delta \in (0 \,..\, \varepsilon)$ such that $\hat{\phi} \neq 0$ on all of δC. Let $\eta \in \widehat{C_c^\infty}$ satisfy $\operatorname{supp} \hat{\eta} \subset \delta C$ and $\hat{\eta} = 1$ on $\frac{1}{2} \delta C$. Then the hypothesis of Proposition 2.8 is satisfied and the estimate (1) reduces to

$$\operatorname{dist}(\eta, S_1^h(\phi); \mathscr{L}_{\bar{p}}) \leqslant \left\| \left(\hat{\phi}(h \cdot) \sum_{j \in \mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\cdot + 2\pi j/h)}{\hat{\phi}(h \cdot + 2\pi j)} \right)^\vee \right\|_{\mathscr{L}_{\bar{p}}} =: \Gamma(h).$$

In view of Theorem 2.3, in order to prove Theorems 7.3 and 7.4, it suffices to show that

$$\Gamma(h) = O(h^k) \qquad \text{as} \quad h \to 0. \tag{7.5}$$

Let $\sigma \in C_c^\infty$ satisfy $\operatorname{supp} \sigma \subset \delta C$ and $\sigma = 1$ on $\operatorname{supp} \hat{\eta}$.

Claim 7.6. If $\bar{p} = 1$ then

$$\Gamma(h) \leqslant \operatorname{const}(d, \eta, \phi) \left\| \left(\hat{\phi}(h \cdot) \sum_{j \in \mathbb{Z}^d \setminus 0} \sigma(\cdot + 2\pi j/h) \right)^\vee \right\|_{L_1} \qquad \forall h \in (0 \,..\, \tfrac{1}{2}).$$

Proof. Fix $h \in (0 \,..\, \frac{1}{2})$, and define

$$\tau := \left(\frac{\hat{\eta}}{\hat{\phi}(h \cdot)} \right)^\vee$$

$$\psi := \left(\hat{\phi} \sum_{j \in \mathbb{Z}^d \setminus 0} \sigma(h^{-1}(\cdot + 2\pi j)) \right)^\vee.$$

For the purpose of proving this claim, there is no loss of generality in assuming that $\psi \in L_1$. Now,

$$\begin{aligned}
\Gamma(h) &= \left\| \left(\hat{\phi}(h \cdot) \sum_{j \in \mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\cdot + 2\pi j/h)}{\hat{\phi}(h \cdot + 2\pi j)} \right)^\vee \right\|_{L_1} \\
&= \left\| \left(\hat{\phi}(h \cdot) \sum_{j \in \mathbb{Z}^d \setminus 0} \sigma(\cdot + 2\pi j/h) \sum_{j \in \mathbb{Z}^d} \frac{\hat{\eta}(\cdot + 2\pi j/h)}{\hat{\phi}(h \cdot + 2\pi j)} \right)^\vee \right\|_{L_1} \\
&= \|\psi *_h' \tau\|_{L_1}, \qquad \text{by Lemma 5.10,} \\
&\leqslant \|\psi\|_{L_1} h^d \|\tau\|_{l_1(h\mathbb{Z}^d)}, \qquad \text{by Lemma 5.1,} \\
&\leqslant \operatorname{const}(d) \|\psi\|_{L_1} \|\tau\|_{L_1}, \qquad \text{by Lemma 5.6,} \\
&= \operatorname{const}(d) \|h^{-d} \psi(\cdot/h)\|_{L_1} \|\tau\|_{L_1} \\
&= \operatorname{const}(d) \left\| \left(\hat{\phi}(h \cdot) \sum_{j \in \mathbb{Z}^d \setminus 0} \sigma(\cdot + 2\pi j/h) \right)^\vee \right\|_{L_1} \left\| \left(\frac{\hat{\eta}}{\hat{\phi}(h \cdot)} \right)^\vee \right\|_{L_1}.
\end{aligned}$$

Note that since $h \in (0 \,..\, \tfrac{1}{2})$,

$$\left\| \left(\frac{\hat{\eta}}{\hat{\phi}(h\cdot)} \right)^{\vee} \right\|_{L_1} = \left\| \left(\frac{\hat{\eta}(h\cdot)\,\hat{\eta}}{\hat{\phi}(h\cdot)} \right)^{\vee} \right\|_{L_1} \leqslant \|\eta\|_{L_1} \left\| \left(\frac{\hat{\eta}(h\cdot)}{\hat{\phi}(h\cdot)} \right)^{\vee} \right\|_{L_1}$$

$$= \|\eta\|_{L_1} \left\| \left(\frac{\hat{\eta}}{\hat{\phi}} \right)^{\vee} \right\|_{L_1} < \infty \qquad \text{by Wiener's lemma.}$$

Therefore,

$$\Gamma(h) \leqslant \operatorname{const}(d, \eta)\, \|\eta\|_{L_1} \left\| \left(\frac{\hat{\eta}}{\hat{\phi}} \right)^{\vee} \right\|_{L_1} \left\| \left(\hat{\phi}(h\cdot) \sum_{j \in \mathbb{Z}^d \backslash 0} \sigma(\cdot + 2\pi j/h) \right)^{\vee} \right\|_{L_1}$$

$$= \operatorname{const}(d, \eta, \phi) \left\| \left(\hat{\phi}(h\cdot) \sum_{j \in \mathbb{Z}^d \backslash 0} \sigma(\cdot + 2\pi j/h) \right)^{\vee} \right\|_{L_1},$$

thus proving the claim.

Claim 7.7.

$$\Gamma(h) \leqslant \operatorname{const}(d, \bar{p}, \phi, \eta, \sigma)\, \|\hat{\phi}(h\cdot)\|_{W^m_{\bar{q}}(\delta C + h^{-1} 2\pi \mathbb{Z}^d \backslash 0)} \qquad \forall h \in (0 \,..\, \tfrac{1}{2}).$$

Proof. Fix $h \in (0 \,..\, \tfrac{1}{2})$.

Case 1. $\bar{p} = 1$.

By Claim 7.6,

$$\Gamma(h) \leqslant \operatorname{const}(d, \eta, \phi) \left\| \left(\hat{\phi}(h\cdot) \sum_{j \in \mathbb{Z}^d \backslash 0} \sigma(\cdot + 2\pi j/h) \right)^{\vee} \right\|_{L_1}$$

$$\leqslant \operatorname{const}(d, \eta, \phi) \left\| \hat{\phi}(h\cdot) \sum_{j \in \mathbb{Z}^d \backslash 0} \sigma(\cdot + 2\pi j/h) \right\|_{W^m_2}, \qquad \text{by Lemma 5.2,}$$

$$\leqslant \operatorname{const}(d, \eta, \phi) \left\| \sum_{j \in \mathbb{Z}^d \backslash 0} \sigma(\cdot + 2\pi j/h) \right\|_{W^m_\infty} \|\hat{\phi}(h\cdot)\|_{W^m_2(\delta C + h^{-1} 2\pi \mathbb{Z}^d \backslash 0)},$$

since $\operatorname{supp} \sigma \subset \delta C$,

$$= \operatorname{const}(d, \eta, \phi, \sigma)\, \|\hat{\phi}(h\cdot)\|_{W^m_2(\delta C + h^{-1} 2\pi \mathbb{Z}^d \backslash 0)}, \qquad \text{since} \quad \sigma \in C^\infty_c.$$

Case 2. $2 \leqslant \bar{p} \leqslant \infty$.

Recall that in this case we assume that $\hat{\phi} \in C^m(\delta \bar{C})$. Hence,

$$\left\| \sum_{j \in \mathbb{Z}^d \backslash 0} \frac{\hat{\eta}(\cdot + 2\pi j/h)}{\hat{\phi}(h\cdot + 2\pi j)} \right\|_{W^m_\infty} = \left\| \frac{\hat{\eta}}{\hat{\phi}(h\cdot)} \right\|_{W^m_\infty} \leqslant \operatorname{const}(d, \eta, \phi).$$

Thus,

$$\begin{aligned}
\Gamma(h) &= \left\| \left(\hat{\phi}(h\cdot) \sum_{j\in\mathbb{Z}^d\setminus 0} \frac{\hat{\eta}(\cdot+2\pi j/h)}{\hat{\phi}(h\cdot+2\pi j)} \right)^{\vee} \right\|_{\mathscr{L}_{\bar{p}}} \\
&\leqslant \operatorname{const}(d,\bar{p}) \left\| \hat{\phi}(h\cdot) \sum_{j\in\mathbb{Z}^d\setminus 0} \frac{\hat{\eta}(\cdot+2\pi j/h)}{\hat{\phi}(h\cdot+2\pi j)} \right\|_{W^m_{\bar{q}}}, \qquad \text{by Lemma 5.2,} \\
&\leqslant \operatorname{const}(d,\bar{p}) \left\| \sum_{j\in\mathbb{Z}^d\setminus 0} \frac{\hat{\eta}(\cdot+2\pi j/h)}{\hat{\phi}(h\cdot+2\pi j)} \right\|_{W^m_\infty} \|\hat{\phi}(h\cdot)\|_{W^m_{\bar{q}}(\delta C+h^{-1}2\pi\mathbb{Z}^d\setminus 0)} \\
&\leqslant \operatorname{const}(d,\bar{p},\eta,\phi)\, \|\hat{\phi}(h\cdot)\|_{W^m_{\bar{q}}(\delta C+h^{-1}2\pi\mathbb{Z}^d\setminus 0)},
\end{aligned}$$

thus completing the proof of the claim.

Therefore, with (7.5) and Claim 7.7 in view, in order to prove the theorems, it suffices to show that

$$\|\hat{\phi}(h\cdot)\|_{W^m_{\bar{q}}(\delta C+h^{-1}2\pi\mathbb{Z}^d\setminus 0)} = O(h^k) \qquad \text{as} \quad h\to 0. \tag{7.8}$$

Following [4], note that since $\rho \geqslant k + d/\bar{q}$, with equality only if $\bar{q}=1$, it follows by the Sobolev imbedding theorem (cf. [1, pp. 97, 217]) that $W^\rho_{\bar{q}}(\delta C)$ is continuously imbedded in $C^k(\delta C)$ (the latter being taken as a closed subspace of $W^k_\infty(\delta C)$). Hence, since $\hat{\phi}(\cdot+2\pi j)\in W^\rho_{\bar{q}}(\delta C)$ we have $\hat{\phi}(\cdot+2\pi j)\in C^k(\delta C)$, $\forall j\in\mathbb{Z}^d\setminus 0$, and

$$\max_{|\beta|\leqslant k} \|(D^\beta\hat{\phi})(\cdot+2\pi j)\|_{L_\infty(\delta C)} \leqslant \operatorname{const}(d,\rho,k,\delta)\, \|\hat{\phi}(\cdot+2\pi j)\|_{W^\rho_{\bar{q}}(\delta C)}, \qquad \forall j\in\mathbb{Z}^d\setminus 0. \tag{7.9}$$

Thus the Strang–Fix conditions of order k are meaningful and as a consequence of their being satisfied we have

$$\begin{aligned}
|(D^\alpha\hat{\phi})(x+2\pi j)| &\leqslant \operatorname{const}(d,k)\, |x|^{k-|\alpha|} \max_{|\beta|=k} \|(D^\beta\hat{\phi})(\cdot+2\pi j)\|_{L_\infty(\delta C)} \\
&\leqslant \operatorname{const}(d,\rho,k,\delta)\, |x|^{k-|\alpha|}\, \|\hat{\phi}(\cdot+2\pi j)\|_{W^\rho_{\bar{q}}(\delta C)} \\
&\qquad \forall x\in\delta C, \qquad j\in\mathbb{Z}^d\setminus 0, \qquad |\alpha|\leqslant k,
\end{aligned} \tag{7.10}$$

where the last inequality follows from (7.9).

Claim 7.11.

$$\|\hat{\phi}(h\cdot)\|_{W^m_{\bar{q}}(\delta C+h^{-1}2\pi j)} \leqslant \operatorname{const}(d,\bar{p},\rho,k,\delta)\, h^k\, \|\hat{\phi}\|_{W^\rho_{\bar{q}}(\delta C+2\pi j)} \qquad \forall h\in(0\,..\,\tfrac{1}{2}), \qquad j\in\mathbb{Z}^d\setminus 0.$$

Proof. Let $j \in \mathbb{Z}^d \backslash 0$, $h \in (0 \,..\, \frac{1}{2})$. Then

$$\begin{aligned}\|\hat{\phi}(h\cdot)\|_{W_{\bar{q}}^m(\delta C + h^{-1}2\pi j)} &= \|\hat{\phi}(h\cdot + 2\pi j)\|_{W_{\bar{q}}^m(\delta C)} \\ &= \left(\sum_{|\alpha| \leqslant m} \|D^\alpha(\hat{\phi}(h\cdot + 2\pi j))\|_{L_{\bar{q}}(\delta C)}^{\bar{q}} \right)^{1/\bar{q}} \\ &= \left(\sum_{|\alpha| \leqslant m} h^{|\alpha|\,\bar{q}} \, \|(D^\alpha\hat{\phi})(h\cdot + 2\pi j)\|_{L_{\bar{q}}(\delta C)}^{\bar{q}} \right)^{1/\bar{q}}.\end{aligned}$$

Hence, in order to prove the claim, it suffices to show that

$$h^{|\alpha|} \, \|(D^\alpha\hat{\phi})(h\cdot + 2\pi j)\|_{L_{\bar{q}}(\delta C)} \leqslant \text{const}(d, \bar{p}, \rho, k, \delta)\, h^k \, \|\hat{\phi}\|_{W_{\bar{q}}^\rho(\delta C + 2\pi j)}, \tag{7.12}$$

for all $|\alpha| \leqslant m$. For that, let $|\alpha| \leqslant m$.

Case 1. $|\alpha| \leqslant k$.

Applying (7.10) to the left side of (7.12) yields

$$\begin{aligned}h^{|\alpha|} \, \|(D^\alpha\hat{\phi})(h\cdot + 2\pi j)\|_{L_{\bar{q}}(\delta C)} &\leqslant h^{|\alpha|} \,\text{const}(d, \rho, k, \delta)\, h^{k-|\alpha|} \\ &\quad \times \|\hat{\phi}(\cdot + 2\pi j)\|_{W_{\bar{q}}^\rho(\delta C)} \, \||\cdot|^{k-|\alpha|}\|_{L_{\bar{q}}(\delta C)} \\ &\leqslant \text{const}(d, \bar{p}, \rho, k, \delta)\, h^k \, \|\hat{\phi}\|_{W_{\bar{q}}^\rho(\delta C + 2\pi j)}.\end{aligned}$$

Therefore, (7.12) holds.

Case 2. $|\alpha| > k$.

Assume without loss of generality that $\rho < k + d/\bar{q} + 1$. Put $q := d/(|\alpha| - k)$. Note that

$$\infty > q \geqslant \frac{d}{m-k} \geqslant \frac{d}{d/\bar{q} + 1 - k} \geqslant \frac{d}{d/\bar{q}} = \bar{q};$$

$$q = \frac{d}{|\alpha| - k} \leqslant \frac{d}{|\alpha| - (\rho - d/\bar{q})} = \frac{d\bar{q}}{d - (\rho - |\alpha|)\,\bar{q}};$$

$$\rho - d/\bar{q} + d/q = \rho - d/\bar{q} + |\alpha| - k \geqslant |\alpha|;$$

$$\rho \geqslant |\alpha| \qquad \text{with equality only if} \quad q = \bar{q}.$$

Hence, by the Sobolev imbedding theorem (cf. [1, pp. 97, 218]), $W_{\bar{q}}^\rho(\delta C)$ is continuously imbedded in $W_q^{|\alpha|}(\delta C)$. In particular,

$$\|D^\alpha g\|_{L_q(\delta C)} \leqslant \text{const}(d, \bar{p}, \rho, k, \delta)\, \|g\|_{W_{\bar{q}}^\rho(\delta C)}, \qquad \forall g \in W_{\bar{q}}^\rho(\delta C). \tag{7.13}$$

Put $r = q/\bar{q}$ and let r' denote the conjugate exponent of r (i.e., satisfying $1/r + 1/r' = 1$). Then,

$$
\begin{aligned}
&(h^{|\alpha|} \, \|(D^\alpha \hat{\phi})(h\cdot + 2\pi j)\|_{L_{\bar{q}}(\delta C)})^{\bar{q}} \\
&\quad = h^{|\alpha| \, \bar{q}} \int_{\delta C} |(D^\alpha \hat{\phi})(hx + 2\pi j)|^{\bar{q}} \, dx \\
&\quad = h^{|\alpha| \, \bar{q} - d} \int_{h\delta C} |(D^\alpha \hat{\phi})(x + 2\pi j)|^{\bar{q}} \, dx \\
&\quad \leqslant h^{|\alpha| \, \bar{q} - d} \, \| |(D^\alpha \hat{\phi})(\cdot + 2\pi j)|^{\bar{q}} \|_{L_r(h\,\delta C)} \, \|1\|_{L_{r'}(h\,\delta C)}, \\
&\qquad\qquad \text{by Hölder's inequality,} \\
&\quad = h^{|\alpha| \, \bar{q} - d} \, \|(D^\alpha \hat{\phi})(\cdot + 2\pi j)\|^{\bar{q}}_{L_q(h\,\delta C)} \, (h\delta)^{d/r'} \\
&\quad \leqslant \text{const}(d, \bar{p}, \rho, k, \delta) \, h^{|\alpha| \, \bar{q} - d + d/r'} \, \|(D^\alpha \hat{\phi})(\cdot + 2\pi j)\|^{\bar{q}}_{L_q(\delta C)} \\
&\quad \leqslant \text{const}(d, \bar{p}, \rho, k, \delta) \, h^{|\alpha| \, \bar{q} - d + d/r'} \, \|\hat{\phi}(\cdot + 2\pi j)\|^{\bar{q}}_{W^\rho_{\bar{q}}(\delta C)} \qquad \text{by (7.13),} \\
&\quad = \text{const}(d, \bar{p}, \rho, k, \delta) \, h^{k\bar{q}} \, \|\hat{\phi}\|^{\bar{q}}_{W^\rho_{\bar{q}}(\delta C + 2\pi j)}.
\end{aligned}
$$

Therefore, (7.12) holds, and hence the claim.

Now,

$$
\begin{aligned}
\|\hat{\phi}(\cdot)\|^{\bar{q}}_{W^m_{\bar{q}}(\delta C + h^{-1} 2\pi \mathbb{Z}^d \backslash 0)} &= \sum_{j \in \mathbb{Z}^d \backslash 0} \|\hat{\phi}(h\cdot)\|^{\bar{q}}_{W^m_{\bar{q}}(\delta C + h^{-1} 2\pi j)} \\
&\leqslant \text{const}(d, \bar{p}, \rho, k, \delta) \, h^{k\bar{q}} \sum_{j \in \mathbb{Z}^d \backslash 0} \|\hat{\phi}\|^{\bar{q}}_{W^\rho_{\bar{q}}(\delta C + 2\pi j)}, \\
&\qquad \text{by Claim 7.11,} \\
&\leqslant \text{const}(d, \bar{p}, \rho, k, \delta) \, h^{k\bar{q}} \, \|\hat{\phi}\|^{\bar{q}}_{W^\rho_{\bar{q}}(\delta C + 2\pi \mathbb{Z}^d \backslash 0)},
\end{aligned}
$$

which, in view of (7.8), proves the theorems. ∎

8. PROOF OF PROPOSITION 3.7

The notation used in the following lemma is of course a silly abstraction of the hypothesis of Proposition 3.7; it serves simply to disarm the usual d-tuple representation of $\mathbb{R}^d$ which, in the present situation, only gets in the way.

LEMMA 8.1. *Let X be a d-dimensional Hilbert Space over $\mathbb{R}$. Let Ξ be a finite multiset of linear functionals defined on X, and suppose that*

$$\#\{\xi \in \Xi \colon \xi \cdot x \neq 0\} \geqslant 2, \qquad \forall x \in X \backslash 0. \tag{8.2}$$

Then there exists $\varepsilon \in (0\,..\,1)$ *such that*

$$\int_X \prod_{\xi \in \Xi} \frac{1}{1+|\xi \cdot x|^{1-\varepsilon}}\, dm_d(x) < \infty,$$

where m_d *is d-dimensional Lesbegue measure on X.*

Proof. For $\varepsilon \in (0\,..\,1)$ define

$$f_\varepsilon(x; \Xi) := \prod_{\xi \in \Xi} \frac{1}{1+|\xi \cdot x|^{1-\varepsilon}}, \qquad x \in X.$$

The lemma is true when $d = 1$ because in that case,

$$\int_X f_{1/4}(x; \Xi)\, dm_1(x) \leqslant \mathrm{const}(\Xi) \int_0^\infty \frac{dt}{1+t^{3/2}} < \infty.$$

Proceeding by induction, assume that the lemma is true whenever $1 \leqslant d \leqslant d'$. Consider $d = d' + 1$. WLOG we may assume that $0 \notin \Xi$. We define the following two sets:

$$\mathbb{H} := \{\xi\bot : \xi \in \Xi\};$$
$$X_0 := \{x \in X : \xi \cdot x \neq 0 \text{ for all } \xi \in \Xi\}.$$

Note that $\mathbb{H}$ is a finite collection of hyperplanes and $X_0 = X \backslash \bigcup \mathbb{H}$. Hence $m_d(X \backslash X_0) = 0$. Now X_0 can be partitioned into finitely many open cells via the equivalence relation

$$x \sim y \qquad \text{if} \quad (\xi \cdot x)(\xi \cdot y) > 0 \qquad \forall \xi \in \Xi. \tag{8.3}$$

Let $\mathbf{\Omega}$ be the collection of these cells. Since $\#\mathbf{\Omega} < \infty$, in order to prove the lemma, it suffices to show that for all $\Omega \in \mathbf{\Omega}$ there exists $\varepsilon \in (0\,..\,1)$ such that

$$\int_\Omega f_\varepsilon(x; \Xi)\, dm_d(x) < \infty. \tag{8.4}$$

So let $\Omega \in \mathbf{\Omega}$. Fix $\tau \in \Omega$.

Claim 8.5.

$$\Omega \subset \bigcup_{H \in \mathbb{H}} \bigcup_{t>0} (t\tau + (\partial\Omega \cap H)).$$

Proof. Let $x \in \Omega$. Since $\tau \in \Omega$ and with (8.3) in view, it follows that for each $\xi \in \Xi$, there exists $t_\xi > 0$ such that $\xi \cdot (x - t_\xi \tau) = 0$. Letting $t := \min_{\xi \in \Xi} t_\xi$, it is easy to see that $x - t\tau \in \partial\Omega \cap H$ for some $H \in \mathbb{H}$, thus proving the claim.

Since $\#\mathbb{H} < \infty$ and with (8.4) and Claim 8.5 in view, in order to prove the lemma, it suffices to show that for all $H \in \mathbb{H}$ there exists $\varepsilon \in (0\,..\,1)$ such that

$$\int_{\bigcup_{t>0} t\tau + (\partial\Omega \cap H)} f_\varepsilon(x; \Xi)\, dm_d(x) < \infty. \tag{8.6}$$

Let $H \in \mathbb{N}$. Note that H is a $d-1$-dimensional Hilbert space and that

$$\#\{\xi \in \Xi\colon \xi \cdot x \neq 0\} \geqslant 2, \qquad \forall x \in H \backslash 0.$$

Therefore by the induction hypothesis, there exists $\varepsilon \in (0\,..\,\frac{1}{3})$ such that

$$\int_H f_{3\varepsilon}(x; \Xi)\, dm_{d-1}(x) < \infty. \tag{8.7}$$

Since $X = H \oplus \operatorname{span}\{\tau\}$, it follows by Fubini's theorem that

$$\begin{aligned}
&\int_{\bigcup_{t>0} t\tau + (\partial\Omega \cap H)} f_\varepsilon(x; \Xi)\, dm_d(x) \\
&\qquad \leqslant \operatorname{const}(H, \tau) \int_0^\infty \int_{t\tau + (\partial\Omega \cap H)} f_\varepsilon(x; \Xi)\, dm_{d-1}(x)\, dt \\
&\qquad = \operatorname{const}(H, \tau) \int_0^\infty \int_{\partial\Omega \cap H} f_\varepsilon(x + t\tau; \Xi)\, dm_{d-1}(x)\, dt.
\end{aligned} \tag{8.8}$$

Note that if $x \in \partial\Omega$, then $(\xi \cdot x)(\xi \cdot \tau) \geqslant 0$ for all $\xi \in \Xi$. Hence,

$$|\xi \cdot (x + t\tau)| = |\xi \cdot x| + |\xi \cdot t\tau|, \qquad \forall x \in \partial\Omega, \quad t \geqslant 0, \quad \xi \in \Xi. \tag{8.9}$$

By (8.2) and the definition of $\mathbb{H}$, there exist $\xi_0, \xi_1 \in \Xi$, distinct in the multiset sense, such that $\xi_0 \perp = H$ and $\xi_1 \cdot \tau \neq 0$.

We wish now to use the following inequality which can be derived simply by considering separately the cases $s + t \geqslant 1$ and $s + t < 1$. If $\alpha, \beta \geqslant 0$, then

$$\frac{1}{1 + (s + t)^{\alpha + \beta}} \leqslant \frac{3}{(1 + s^\alpha)(1 + t^\beta)}, \qquad \forall s, t \geqslant 0. \tag{8.10}$$

We will apply this inequality with $\alpha = 2\varepsilon$ and $\beta = 1 - 3\varepsilon$ which is valid since $\varepsilon \in (0\,..\,\frac{1}{3})$. Now, for $t > 0$ and $x \in \partial\Omega \cap H$,

$$\begin{aligned}
f_\varepsilon(t\tau+x;\Xi) &= \prod_{\xi\in\Xi}\frac{1}{1+(|\xi\cdot t\tau|+|\xi\cdot x|)^{1-\varepsilon}}, \qquad \text{by (8.9)}\\
&= \frac{1}{1+|\xi_0\cdot t\tau|^{1-\varepsilon}}\prod_{\xi\in\Xi\backslash\xi_0}\frac{1}{1+(|\xi\cdot t\tau|+|\xi\cdot x|)^{1-\varepsilon}},\\
&\qquad \text{since} \quad \xi_0\cdot x=0,\\
&\leqslant \frac{1}{1+|\xi_0\cdot t\tau|^{1-\varepsilon}}\prod_{\xi\in\Xi\backslash\xi_0}\frac{3}{(1+|\xi\cdot t\tau|^{2\varepsilon})(1+|\xi\cdot x|^{1-\varepsilon})}, \quad \text{by (8.10)},\\
&\leqslant \frac{1}{(1+|\xi_0\cdot t\tau|^{1-\varepsilon})(1+|\xi_1\cdot t\tau|^{2\varepsilon})}\prod_{\xi\in\Xi\backslash\xi_0}\frac{3}{1+|\xi\cdot x|^{1-3\varepsilon}}\\
&= \frac{1/3}{(1+|\xi_0\cdot t\tau|^{1-\varepsilon})(1+|\xi_1\cdot t\tau|^{2\varepsilon})}\prod_{\xi\in\Xi}\frac{3}{1+|\xi\cdot x|^{1-3\varepsilon}},\\
&\qquad \text{since} \quad \xi_0\cdot x=0,\\
&\leqslant \text{const}(\Xi,\tau,\xi_0,\ \xi_1)\,\frac{1}{1+t^{1+\varepsilon}}\,f_{3\varepsilon}(x;\Xi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^\infty\int_{\partial\Omega\cap H} f_\varepsilon(t\tau+x;\Xi)\,dm_{d-1}(x)\,dt\\
&\quad\leqslant \text{const}(\Xi,\tau,\xi_0,\xi_1)\int_0^\infty\frac{1}{1+t^{1+\varepsilon}}\int_{\partial\Omega\cap H} f_{3\varepsilon}(x;\Xi)\,dm_{d-1}(x)\,dt<\infty,\\
&\qquad \text{by (8.7)}
\end{aligned}$$

Thus, in view of (8.8) and (8.6), the lemma is proved. ∎

LEMMA 8.11. *For all* $x\in\mathbb{R}^d$,

$$\inf\{|nx-j| : n\in\mathbb{N} \text{ and } j\in\mathbb{Z}^d\}=0.$$

Proof. For $x\in\mathbb{R}^d$, let $C(x)$ be the unique element of $[-\frac{1}{2}..\frac{1}{2})^d$ such that $x-C(x)\in\mathbb{Z}^d$. Note that

$$|C(x)|=\inf\{|x-j| : j\in\mathbb{Z}^d\}, \qquad \forall x\in\mathbb{R}^d.$$

Fix $x\in\mathbb{R}^d$ and let $\varepsilon>0$. Since $C(nx)\in[-\frac{1}{2}..\frac{1}{2}]^d$ for all $n\in\mathbb{N}$, there exists $y\in[-\frac{1}{2}..\frac{1}{2}]^d$ and $m,n\in\mathbb{N}$ with $m<n$ such that

$$|C(mx)-y|+|C(nx)-y|<\varepsilon.$$

Hence,

$$|C((n-m)\,x)| \leqslant |C(mx) - C(nx)| \leqslant |C(mx) - y| + |C(nx) - y| < \varepsilon. \quad ∎$$

Proof of Proposition 3.7. Assume that $k'(\Xi) \geqslant 2$. Put $R := \max_{\xi \in \Xi} |\xi|$ and $\delta := 1/(2R)$. We will show first of all that

$$\#\{\xi \in \Xi: \zeta \cdot x \neq 0\} \geqslant 2, \qquad \forall x \in \mathbb{R}^d \backslash 0. \tag{8.12}$$

Let $x \in \mathbb{R}^d \backslash 0$. By Lemma 8.11, there exists $n \in \mathbb{N}$ and $j \in \mathbb{Z}^d \backslash 0$ such that $|nx - j| < \delta$. Now if $\xi \in K_j$, then $|\xi \cdot j| \geqslant 1$. Hence,

$$|\xi \cdot nx| \geqslant |\xi \cdot j| - |\xi \cdot (nx - j)| \geqslant 1 - |\xi|\, |nx - j| \geqslant \tfrac{1}{2}.$$

Therefore,

$$\#\{\xi \in \Xi: \xi \cdot x \neq 0\} \geqslant \# K_j \geqslant 2,$$

and hence (8.12). Therefore by Lemma 8.1,

$$\int_{\mathbb{R}^d} \prod_{\xi \in \Xi} \frac{dx}{1 + |\xi \cdot x|} < \infty. \tag{8.13}$$

Now, if $j \in \mathbb{Z}^d \backslash 0$ and $x \in j + \delta B$, then

$$\begin{aligned} \prod_{\xi \in \Xi} \frac{1}{1 + |\xi \cdot j|} &\leqslant \prod_{\xi \in \Xi} \frac{1}{1 + |\xi \cdot x| - |\xi \cdot (x - j)|} \leqslant \prod_{\xi \in \Xi} \frac{1}{1/2 + |\xi \cdot x|} \\ &\leqslant \prod_{\xi \in \Xi} \frac{2}{1 + |\xi \cdot x|}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{j \in \mathbb{Z}^d \backslash 0} \prod_{\xi \in \Xi} \frac{1}{1 + |\xi \cdot j|} &\leqslant \frac{1}{m_d(\delta B)} \sum_{j \in \mathbb{Z}^d \backslash 0} \int_{j + \delta B} \prod_{\xi \in \Xi} \frac{2}{1 + |\xi \cdot x|}\, dx \\ &\leqslant \text{const}(d, \delta, \Xi) \int_{\mathbb{R}^d} \prod_{\xi \in \Xi} \frac{1}{1 + |\xi \cdot x|}\, dx < \infty \end{aligned}$$

by (8.13). ∎

ACKNOWLEDGMENTS

I am honored to thank Amos Ron for various insights into some of the important issues of this area which he has offered during our many discussions, and particularly for the suggestions which prompted the discovery of Theorem 7.3. For his help with Definition 2.2, I am pleased to thank Ron DeVore. I also thank Carl de Boor and George Kyriazis for their pertinent comments.

REFERENCES

1. R. A. Adams, “Sobolev Spaces,” Academic Press, New York, 1975.
2. I. Babauška, Approximation by Hill functions, *Comment. Math. Univ. Carolin.* **11** (1970), 787–811.
3. R. K. Beatson and W. A. Light, Quasi-interpolation in the absence of polynomial reproduction, *in* “Numerical Methods of Approximation Theory” (D. Braess and L. L. Schumaker, Eds.), pp. 21–39, Birkhäuser-Verlag, Basel, 1992.
4. C. de Boor, R. A. DeVore, and A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$, *Trans. Amer. Math. Soc.*, to appear.
5. C. de Boor and K. Höllig, *B*-splines from parallelepipeds, *J. Analyse Math.* **42** (1982/1983), 99–115.
6. C. de Boor, K. Höllig, and S. Riemenschneider, “Box Splines,” Springer-Verlag, New York, 1993.
7. C. de Boor and R.-Q. Jia, Controlled approximation and a characterization of the local approximation order, *Proc. Amer. Math. Soc.* **95** (1985), 547–553.
8. C. de Boor and A. Ron, Fourier analysis of approximation orders from principal shift-invariant spaces, *Constr. Approx.* **8** (1992), 427–462.
9. C. de Boor and A. Ron, The least solution for the polynomial interpolation problem, *Math. Z.* **210** (1992), 347–378.
10. M. D. Buhmann, Multivariate interpolation in odd-dimensional Euclidean spaces using multiquadrics, *Constr. Approx.* **6** (1990), 21–34.
11. W. Dahmen and C. A. Micchelli, On the approximation order from certain multivariate spline spaces, *J. Austral. Math. Soc. Ser B* **26** (1984), 233–246.
12. F. Di Guglielmo, Méthod des éléments finis: Une famille d’approximations des espaces de Sobolev par les translatés de *p* functions, *Calcolo* **7** (1970), 185–233.
13. N. Dyn and A. Ron, Local approximation by certain spaces of multivariate exponential-polynomials, approximation order of exponential box splines and related interpolation problems, *Trans. Amer. Math. Soc.* **319** (1990), 381–404.
14. H. G. Feichtinger and K. Gröchenig, Irregular sampling theorems and series expansions of band-limited functions, *J. Math. Anal. Appl.* **167** (1992), 530–556.
15. E. J. Halton and W. A. Light, On local and controlled approximation order, *J. Approx. Theory* **72**, No. 3 (1993), 268–277.
16. I. R. H. Jackson, An order of convergence for some radial basis functions, *IMA J. Numer. Anal.* **9** (1989), 567–587.
17. R.-Q. Jia, A counterexample to a result concerning controlled approximation, *Proc. Amer. Math. Soc.* **97** (1986), 199–212.
18. R.-Q. Jia and J. Lei, Approximation by multiinteger translates of functions having global support, *J. Approx. Theory* **72** (1993), 2–23.
19. R.-Q. Jia and C. A. Micchelli, Using the refinement equation for the construction of pre-wavelets. II. Powers of two, *in* “Curves and Surfaces” (P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, Eds.), pp. 209–246, Academic Press, New York, 1991.
20. M. J. Johnson, An upper bound on the approximation power of principal shift-invariant spaces, *Constr. Approx.*, to appear.
21. G. C. Kyriazis, Approximation from shift-invariant spaces, *Constr. Approx.*, to appear.
22. G. C. Kyriazis, Approximation orders of principal shift-invariant spaces generated by box splines, preprint.
23. Y. Katzneslon, “An Introduction to Harmic Analysis,” Wiley, New York, 1968.
24. J. Lei and R.-Q. Jia, Approximation by piecewise exponentials, *SIAM J. Math. Anal* **22** (1991), 1776–1789.

25. W. A. Light and E. W. Cheney, Quasi-interpolation with translates of a function having non-compact support, *Constr. Approx.* **8** (1992), 35–48.
26. J. Peetre, "New Thoughts on Besov Spaces," Math. Dept., Duke Univ., Durham, NC, 1976.
27. A. Ron, Exponential box splines, *Const. Approx.* **4** (1988), 357–378.
28. A. Ron, A characterization of the approximation order of multivariate spline spaces, *Studia Math.* **98**, No. 1 (1991), 73–90.
29. A. Ron, The L_2-approximation orders of principal shift-invariant spaces generated by a radial basis function, *in* "Numerical Methods of Approximation Theory" (D. Braess and L. L. Schumaker, Eds.), Vol. 9, pp. 245–268, Birkhäuser-Verlag, Basel, 1991.
30. A. Ron, "Approximation Orders of and Approximation Maps from Local Principal Shift-Invariant Spaces," CMS TSR No. 92-2, University of Wisconsin–Madison, May 1993.
31. A. Ron and N. Sivakumar, The approximation order of box spline spaces, *Proc. Amer. Math. Soc.*, to appear.
32. W. Rudin, "Functional Analysis," McGraw–Hill, New York, 1973.
33. I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, *A*, *B*, *Quart. Appl. Math.* **4** (1946), 45–99, 112–141.
34. G. Strang and G. Fix, A Fourier analysis of the finite element variational method, *in* "Constructive Aspects of Functional Analysis" (G. Geymonat, Ed.), pp. 793–840, C.I.M.E., 1973.
35. E. M. Stein and G. Weiss, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, NJ, 1971.